Name: Key

M555: Differential Equations I (Spring 2018)
Instructor: Justin Ryan
Good Problems 5: Sections 3.1–3.4

Instructions *Complete all problems, showing enough work. A selection of problems will be graded based on the organization and clarity of the work shown in addition to the final solution (provided one exists).*

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1. Find the general solutions of the differential equations.

a.) $6y'' - y' - y = 0$

$6r^2 - r - 1 = 0$

$(3r+1)(2r-1) = 0$

$r = -1/3 \quad r = 1/2$

$$\boxed{y(t) = C_1 e^{-1/3 t} + C_2 e^{1/2 t}}$$

b.) $y'' - 2y' + 6y = 0$

$r^2 - 2r + 6 = 0$

$(r^2 - 2r + 1) + 6 - 1 = 0$

$(r-1)^2 = -5$

$r - 1 = \pm\sqrt{5}\, i$

$r = 1 \pm i\sqrt{5}$

$$\boxed{y(t) = e^{t}\left(C_1 \cos(\sqrt{5}t) + C_2 \sin(\sqrt{5}\, t)\right)}$$

c.) $y'' - 2y' + y = 0$

$r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$r = 1, 1$

$$\boxed{y(t) = C_1 e^{t} + C_2 t e^{t}}$$

2. Find the particular solutions of the initial value problems.

$$a.)\begin{cases} y''+4y'+5y=0 \\ y(0)=1 \\ y'(0)=0 \end{cases}$$

$r^2+4r+5=0$

$(r^2+4r+4)+1=0$

$(r+2)^2=-1$

$r=-2\pm i$

$y(t)=e^{-2t}(c_1\cos t+c_2\sin t)$

$y'(t)=e^{-2t}(-c_1\sin t+c_2\cos t)-2e^{-2t}(c_1\cos t+c_2\sin t)$

$y(0)=c_1=1$

$y'(0)=c_2-2c_1=0 \rightarrow c_2=2c_1=2$

$\boxed{y(t)=e^{-2t}(\cos t+2\sin t)}$

$$b.)\begin{cases} y''+4y'+3y=0 \\ y(0)=2 \\ y'(0)=-1 \end{cases}$$

$r^2+4r+3=0$

$(r+3)(r+1)=0$

$r=-3,-1$

$y(t)=c_1e^{-3t}+c_2e^{-t}$

$y'(t)=-3c_1e^{-3t}-c_2e^{-t}$

$y(0)=c_1+c_2=2$

$y'(0)=-3c_1-c_2=-1$

$-2c_1=1$ $\quad c_2=2-c_1=2+\tfrac{1}{2}=\tfrac{5}{2}$

$c_1=-\tfrac{1}{2}$

$\boxed{y(t)=-\tfrac{1}{2}e^{-3t}+\tfrac{5}{2}e^{-t}}$

3. Consider the differential equation

$$(x-1)\,y'' - x\,y' + y = 0, \quad x > 0.$$

$x > 1$.

a.) Show that $\varphi_1(x) = e^x$ is a solution.

b.) Use the method of reduction of order to find a second solution of the differential equation. Verify that the second solution is indeed a new solution.

a.) $\varphi_1 = \varphi_1' = \varphi_1'' = e^x$

Plugging in to the DE, $(x-1)e^x - xe^x + e^x = e^x\big((x-1)-x+1\big) = e^x(0) = 0$ ✓

b.) Guess $\varphi(x) = v(x)e^x$

$\varphi'(x) = v'(x)e^x + v(x)e^x$

$\varphi''(x) = v''(x)e^x + v'(x)e^x + v'(x)e^x + v(x)e^x = v''(x)e^x + 2v'(x)e^x + v(x)e^x$

Plug in:

$(x-1)\varphi'' - x\varphi' + \varphi = (x-1)(v''e^x + 2v'e^x + ve^x) - x(v'e^x + ve^x) + ve^x$

$= e^x\Big[(x-1)(v'' + 2v' + v) - x(v' + v) + v\Big]$

$= e^x\left(xv'' + \cancel{2}xv' + \cancel{xv} - v'' - 2v' - \cancel{v} - \cancel{xv'} - \cancel{xv} + \cancel{v}\right)$

$= e^x\big((x-1)v'' + (x-2)v'\big) = 0$

$\Rightarrow \dfrac{d(v')}{dx} = \dfrac{-(x-2)}{(x-1)}v'$

$\Rightarrow \dfrac{1}{v'}d(v') = \dfrac{-(x-2)}{x-1}dx = -\left(\dfrac{x-1}{x-1}\right) - \dfrac{-1}{x-1}dx$

$\ln(v') = -x + \ln(x-1) + C_1$

$v' = C_1 e^{-x}(x-1)$

u	dv
$+\ (x-1)$	e^{-x}
$-\ 1$	$-e^{-x}$
$+\ 0$	e^{-x}

$v = \int C_1 e^{-x}(x-1)\,dx = C_1\left(-(x-1)e^{-x} - e^{-x}\right) + C_2 = C_1 xe^{-x} + C_2.$

Therefore the solution is $\boxed{\varphi(x) = C_1 x + C_2 e^x.}$

The Wronskian is $\begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix} = e^x(x-1) \neq 0$ for $x \neq 1$.

4. Consider the differential equation

$$y'' + 4y = t^2 + 3e^t.$$

a.) Find the solution y_h of the associated homogeneous equation.

b.) Use the method of undetermined coefficients to find the general solution $y = Y + y_h$ of the given DE.

a.) $r^2 + 4 = 0$

$r = \pm 2i$

$y_h(t) = C_1 \cos(2t) + C_2 \sin(2t)$

b.) Guess: $Y(t) = At^2 + Bt + C + De^t$

$Y'(t) = 2At + B + De^t$

$Y''(t) = 2A + De^t$

$Y'' + 4Y = (2A + De^t) + 4(At^2 + Bt + C + De^t)$

$= 4At^2 + 4Bt + (2A + 4C) + 5De^t = t^2 + 3e^t$

t^2: $4A = 1 \rightarrow A = 1/4$

t: $4B = 0 \quad B = 0$

*: $2A + 4C = 0 \quad C = -1/2 A = -1/8$

e^t: $5D = 3 \quad D = 3/5$

The solution is:

$y(t) = y_h(t) + Y(t)$

$$\boxed{y(t) = C_1 \cos(2t) + C_2 \sin(2t) + \frac{1}{4}t^2 - \frac{1}{8} + \frac{3}{5}e^t}$$