Name: Key
M555: Differential Equations I (Spring 2018)
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Good Problems 6: Sections 3.5, 3.6, Chapter 4

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Instructions *Complete all problems, showing enough work. A selection of problems will be graded based on the organization and clarity of the work shown in addition to the final solution (provided one exists).*

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1. Find the general solution of the differential equation.

$$\begin{cases} y'' + y = \tan t \\ 0 < t < \frac{\pi}{2} \end{cases}$$

$r^2 + 1 = 0$

$r = \pm i$

$y_h(t) = C_1 \cos(t) + C_2 \sin(t)$

$W(y_1, y_2) = 1.$

$$y(t) = -\cos t \underbrace{\int \sin t \tan t \, dt}_{I_1} + \sin t \underbrace{\int \cos t \tan t \, dt}_{I_2}$$

$$I_1 = \int \sin t \tan t \, dt = \int \frac{\sin^2 t}{\cos t} dt = \int \frac{1 - \cos^2 t}{\cos t} dt = \int \sec t - \cos t \, dt = \ln|\sec t + \tan t| - \sin t + C_1$$

$$I_2 = \int \cos t \tan t \, dt = \int \sin t \, dt = -\cos t + C_2$$

$$\boxed{y(t) = -\cos t \ln|\sec t + \tan t| + C_1 \cos t + C_2 \sin t}$$

2. Consider the initial value problem (IVP),

$$\begin{cases} t^2 y'' - 7ty' + 7y = 0, \\ y(1) = 1, \\ y'(1) = -1. \end{cases}$$

This is a second order linear homogeneous differential equation with non-constant coefficients. Assume that the solutions are of the form

$$\varphi(t) = t^r$$

for some r. Plug this into the DE and solve for r to find the general solution, then find the particular solution of the IVP.

$\varphi(t) = t^r$

$\varphi'(t) = r t^{r-1}$

$\varphi''(t) = r(r-1)\, t^{r-2}$

$$t^2\varphi'' - 7t\varphi' + 7\varphi = t^2 r(r-1)t^{r-2} - 7t\, r\, t^{r-1} + 7t^r$$
$$= t^r\left(r^2 - r - 7r + 7\right)$$
$$= t^r\left(r^2 - 8r + 7\right) = 0$$

$$(r-7)(r-1) = 0$$

$$r = 7, \quad r = 1$$

So the general solution is $\varphi(t) = c_1 t^7 + c_2 t$.

$\varphi'(t) = 7c_1 t^6 + c_2$

$\varphi(1) = c_1 + c_2 = 1$

$\varphi'(1) = 7c_1 + c_2 = -1$

$-6c_1 = +2$

$c_1 = -1/3$

$c_2 = 1 - c_1 = 1 + 1/3 = 4/3$

The solution of the IVP is

$$\boxed{\varphi(t) = -\frac{1}{3}t^7 + \frac{4}{3}t}$$

3. Consider the second order linear differential equation

$$at^2y'' + bty' + cy = 0.$$

Differential equations of this type are known as *Euler equations.* Apply the method described in the previous good problem to find the general solution.

$$\left.\begin{aligned} \varphi(t) &= t^r \\ \varphi'(t) &= rt^{r-1} \\ \varphi''(t) &= r(r-1)t^{r-2} \end{aligned}\right\} \quad at^2\varphi'' + bt\varphi' + c\varphi = at^2 r(r-1)t^{r-2} + bt\, rt^{r-1} + ct^r$$

$$= t^r\left(ar(r-1) + br + c\right) = 0$$

characteristic polynomial is: $\quad ar^2 + (b-a)r + c = 0$

$$\text{or} \quad r^2 + \left(\frac{b}{a} - 1\right) r + \frac{c}{a} = 0$$

$$r^2 + \left(\frac{b}{a}-1\right)r + \left(\frac{b-a}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b-a}{2a}\right)^2$$

$$\left(r + \left(\frac{b-a}{2a}\right)\right)^2 = \frac{(b-a)^2 - 4ac}{(2a)^2}$$

$$r = \frac{-(b-a)}{2a} \pm \sqrt{\frac{(b-a)^2 - 4ac}{(2a)^2}}$$

$$r = \frac{-(b-a) \pm \sqrt{(b-a)^2 - 4ac}}{2a} = r_1, r_2$$

The general solution is,

$$\underline{\varphi(t) = C_1 t^{r_1} + C_2 t^{r_2}}$$

4. Find the general solution of the differential equation

$$y'' - 4y' + 4y = e^t \sin(t).$$

$r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

$r = 2, 2$

$y_h(t) = c_1 e^{2t} + c_2 t e^{2t}$

Undetermined Coefficients: Guess $Y(t) = e^t(A\cos t + B\sin t) = Ae^t\cos t + Be^t\sin t$

$Y'(t) = Ae^t\cos t - Ae^t\sin t + Be^t\sin t + Be^t\cos t$

$Y'(t) = (A+B)e^t\cos t + (-A+B)e^t\sin t$

$Y''(t) = (A+B)e^t\cos t - (A+B)e^t\sin t + (-A+B)e^t\sin t + (-A+B)e^t\cos t$

$Y''(t) = 2Be^t\cos t - 2Ae^t\sin t$

$Y'' - 4Y' + 4Y = 2Be^t\cos t - 2Ae^t\sin t - 4((A+B)e^t\cos t + (-A+B)e^t\sin t) + 4(Ae^t\cos t + Be^t\sin t)$

$= e^t\cos t(2B - 4A - 4B + 4A) + e^t\sin t(-2A + 4A - 4B + 4B)$

$= 6Be^t\cos t + 2Ae^t\sin t$

$\Rightarrow 6B = 0 \qquad 2A = 1 \qquad \Rightarrow A = 1/2,\ B = 0$

So, $Y(t) = \frac{1}{2}e^t\cos t$

and

$$\boxed{\varphi(t) = y_h(t) + Y(t) = c_1 e^{2t} + c_2 t e^{2t} + \frac{1}{2}e^t\cos t}$$