Name: Key

M555: Differential Equations I (Su.19)
Good Problems 2
Sections 2.6, 2.4, 2.8

Instructions. Complete all problems, showing enough work. All work must be done on this paper. You may use your own hand-written notes, but you may not use any electronic devices.

1. Consider the initial value problem,

$$\begin{cases} y' + y^3 = 0, \\ y(0) = y_0 \end{cases}$$

a.) [15 points] Find the solution of the IVP in terms of y_0.

Separable: $\frac{dy}{dt} = -y^3 \to \int \frac{-1}{y^3}\,dy = \int dt \to \frac{1}{2}\frac{1}{y^2} = t + C \xrightarrow{\text{IV}:(0,y_0)} \frac{1}{2y_0^2} = C$

$\to 2y^2 = \frac{1}{t + \frac{1}{2y_0^2}} \to y^2 = \frac{1}{2t + 1/y_0^2} \to y^2 = \frac{y_0^2}{2ty_0^2 + 1}$

$\to y = \pm\sqrt{\frac{y_0^2}{2ty_0^2+1}}$

sign depends on sign of y_0:
+ if $y_0 > 0$, − if $y_0 < 0$

b.) [10 points] Describe how the domain of definition of the solution function depends on the value of y_0. Be sure to include all possibilities.

Domain: $2ty_0^2 + 1 > 0 \to t > \frac{-1}{2y_0^2}$

If $y_0 = 0$: domain $= \mathbb{R}$

If $y_0 \neq 0$: domain $= \left(\frac{-1}{2y_0^2}, \infty\right)$

2. Consider the differential equation

$$\underbrace{x^2 y^3}_{M} + \underbrace{x(1+y^2)}_{N} y' = 0.$$

a.) [5 points] Show that the DE is not exact as written.

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 3x^2y^2 \\ \frac{\partial N}{\partial x} &= 1+y^2 \end{aligned}\right\} \neq$$

Since $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \neq 0$, the DE is not Exact.

b.) [5 points] Show that the DE becomes exact when using the integrating factor $\mu(x, y) = \frac{1}{xy^3}$.

$$\left.\begin{aligned} \mu M &= \frac{x^2y^3}{xy^3} = x \rightarrow \frac{\partial(\mu M)}{\partial y} = 0 \\ \mu N &= \frac{x(1+y^2)}{xy^3} = \frac{1+y^2}{y^3} = \frac{1}{y^3} + \frac{1}{y} \rightarrow \frac{\partial(\mu N)}{\partial x} = 0 \end{aligned}\right\}$$

Exact!
(actually, separable ☺)

c.) [15 points] Find the general solution to the DE. You do not need to solve for y explicitly.

The DE has become: $x\,dx + \left(\frac{1}{y^3} + \frac{1}{y}\right) dy = 0$

Integrating: $\boxed{\frac{1}{2}x^2 - \frac{1}{2}\frac{1}{y^2} + \ln y = C}$

3. Consider the differential equation

$$\begin{cases} \underbrace{(xy^2 + kx^2 y)}_{M}\, dx + \underbrace{(x+y)\, x^2}_{N}\, dy = 0, \\ y(1) = 1. \end{cases}$$

a.) [10 points] Find the value of k for which the differential equation is exact.

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= 2xy + kx^2 \\ \frac{\partial N}{\partial x} &= 2xy + 3x^2 \end{aligned}\right\} \quad \text{Exact only if } \boxed{k=3}$$

b.) [15 points] Using the value of k that you found in part *a*, solve the initial value problem. You do not need to solve for y explicitly.

$$\left.\begin{aligned} M &= xy^2 + 3x^2y \\ N &= x^3 + x^2y \end{aligned}\right\} \quad \begin{aligned} \varphi &= \int xy^2 + 3x^2y\, dx = \tfrac{1}{2}x^2y^2 + x^3y + \overbrace{C_1(y)}^{=0} \\ \varphi &= \int x^3 + x^2y\, dy = x^3y + \tfrac{1}{2}x^2y^2 + C_2(x) \end{aligned}$$

The general solution is $\varphi(x,y) = \frac{1}{2}x^2y^2 + x^3y = C$

Plugging in the IV, $(1,1)$: $\quad \frac{1}{2}1^2 1^2 + 1^3 1 = \frac{1}{2} + 1 = \frac{3}{2} = C$

so the solution is

$$\frac{1}{2}x^2y^2 + x^3y = \frac{3}{2}$$

or

$$\boxed{x^2y^2 + 2x^3y = 3}$$

4. [7 points] Make a linear change of variables to produce an equivalent differential equation with initial value $(0,0)$.

$$\begin{cases} y' = t^2 + y^2, \\ y(1) = 2. \end{cases}$$

Let: $u = t-1$

$v = y-2$

So: $\left.\begin{array}{l} t = u+1 \\ y = v+2 \\ dt = du \\ dy = dv \end{array}\right\}$ and the IVP becomes $\begin{cases} \frac{dv}{du} = (u+1)^2 + (v+2)^2, \\ v(0) = 0. \end{cases}$

5. [18 points] Consider the initial value problem,

$$\begin{cases} y' = ty + 1, \\ y(0) = 0. \end{cases}$$

Using the initial guess $\varphi_0(t) = 0$, perform Picard's iterative method to find $\varphi_1, \varphi_2, \varphi_3$, and φ_4.

$\varphi_0 = 0$

$\varphi_1 = \int_0^t 0+1 \, du = t$

$\varphi_2 = \int_0^t u^2+1 \, du = \frac{1}{3}t^3 + t$

$\varphi_3 = \int_0^t \frac{1}{3}u^4 + u^2 + 1 \, du = \frac{1}{15}t^5 + \frac{1}{3}t^3 + t$

$\varphi_4 = \int_0^t \frac{1}{15}u^6 + \frac{1}{3}u^4 + u^2 + 1 \, du = \frac{1}{105}t^7 + \frac{1}{15}t^5 + \frac{1}{3}t^3 + t$

So,
$$\boxed{\begin{array}{l} \varphi_0(t) = 0 \\ \varphi_1(t) = t \\ \varphi_2(t) = \tfrac{1}{3}t^3 + t \\ \varphi_3(t) = \tfrac{1}{15}t^5 + \tfrac{1}{3}t^3 + t \\ \varphi_4(t) = \tfrac{1}{105}t^7 + \tfrac{1}{15}t^5 + \tfrac{1}{3}t^3 + t \end{array}}$$

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