Name: Key

M555: Differential Equations I (Su.19)

Good Problems 4

Selections from Chapter 3

Instructions. Complete all problems, showing enough work. All work must be done on this paper. You may use your own hand-written notes, but you may not use any electronic devices.

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1. [15 points] Find the general solution of the Euler equation $t^2y'' - 5ty' + 9y = 0$.

Euler! Char. Eq'n: $r^2 - 6r + 9 = 0$

$(r-3)^2 = 0$

$r = 3, 3$

$$\boxed{y = c_1 t^3 + c_2 t^3 \ln|t|, \quad t \neq 0.}$$

2. [15 points] Compute the Wronskian of the functions t^r and $t^r \ln(t)$ for $t > 0$ and $r \in \mathbb{R}$.

$$W(t^r, t^r \ln t) = \begin{vmatrix} t^r & t^r \ln t \\ rt^{r-1} & rt^{r-1}\ln t - t^{r-1} \end{vmatrix}$$

$= rt\ln t - t - rt\ln t = -t \neq 0$ since $t > 0$.

3. [20 points] Suppose g is an arbitrary continuous function. Write down an integral solution to the non-homogeneous differential equation,

$$y'' + 4y' + 13y = g(t).$$

Homog: Char. Eqn: $r^2+4r+13=0$

$$\left.\begin{aligned} r^2+4r+4&=-9 \\ (r+2)^2&=-3^2 \\ r&=-2\pm 3i \end{aligned}\right\} \quad y_h = C_1 \underbrace{e^{-2t}\cos(3t)}_{y_1} + C_2 \underbrace{e^{-2t}\sin(3t)}_{y_2}.$$

$$W = \left(e^{-2t}\right)^2 \begin{vmatrix} \cos(3t) & \sin(3t) \\ -2\cos(3t)-3\sin(3t) & -2\sin(3t)+3\cos(3t) \end{vmatrix}$$

$$= e^{-4t}\left[\cancel{-2\cos(3t)\sin(3t)} + 3\cos^2(3t) + \cancel{2\cos(3t)\sin(3t)} + 3\sin^2(3t)\right]$$

$$= 3e^{-4t}$$

$$W_1 = -g(t)e^{-2t}\sin(3t)$$

$$W_2 = g(t)e^{-2t}\cos(3t)$$

V.o.P. soln to non-homog. eqn: $\quad y = y_1\int \frac{W_1}{W}\,dt + y_2\int\frac{W_2}{W}\,dt$

$$\boxed{y = \frac{1}{3}e^{-2t}\cos(3t)\int -g(t)e^{2t}\sin(3t)\,dt + \frac{1}{3}e^{-2t}\sin(3t)\int g(t)e^{2t}\cos(3t)\,dt}$$

4. [15 points] You wish to use the method of undetermined coefficients to determine a solution of the non-homogeneous differential equation. Write down a suitable form of the entire solution. Do **NOT** solve for the undetermined coefficients.

$$y'' + 2y' + 2y = 3e^{-t} + 2e^{-t}\cos t + 4e^{-t}t^2 \sin t$$

Homog: $r^2+2r+2=0$

$(r+1)^2=-1$

$r=-1\pm i$

$y_h = C_1 e^{-t}\cos t + C_2 e^{-t}\sin t$

$Y_1 = Ae^{-t}$

$Y_2 = t(Bt^2+Ct+D)e^{-t}\cos t + t(Et^2+Ft+G)e^{-t}\sin t$

So,

$$\boxed{y(t) = Ae^{-t} + (Bt^3+Ct^2+Dt)e^{-t}\cos t + (Et^3+Ft^2+Gt)e^{-t}\sin t + C_1e^{-t}\cos t + C_2 e^{-t}\sin t}$$

5. [15 points] Solve the initial value problem using your favorite method.

$$\begin{cases} y'' + 4y = t^2, \\ y(0) = 0, \quad y'(0) = 2. \end{cases}$$

Homog. $r^2+4=0$

$r=\pm 2i$

$y_h = C_1\cos(2t) + C_2\sin(2t)$

$Y = At^2+Bt+C$

$Y' = 2At+B$

$Y'' = 2A$

$Y''+4Y = 2A + 4At^2 + 4Bt + 4C = t^2$

$4A=1 \to A=1/4$

$4B=0 \to B=0$

$2A+4C=0 \quad C=-1/8$

$y = 1/4 t^2 - 1/8 + C_1\cos(2t) + C_2\sin(2t)$

$y' = \frac{1}{2}t - 2\sin(2t)C_1 + 2\cos(2t)C_2$

$y(0) = -\frac{1}{8} + C_1 = 0 \quad \to C_1 = 1/8$

$y'(0) = 2C_2 = 2 \quad \to C_2 = 1$

$$\boxed{y = \frac{1}{4}t^2 - \frac{1}{8} + \frac{1}{8}\cos(2t) + \sin(2t)}$$

6. [20 points] Consider the differential equation $xy'' - y' + 4x^3 y = 0$, $x > 0$.

a.) Show that $y_1(x) = \sin(x^2)$ is a solution.

$y_1' = 2x\cos(x^2)$

$y_1'' = 2\cos(x^2) - 4x^2\sin(x^2)$

$xy_1'' - y_1' + 4x^3 y = \underline{2x\cos(x^2)} - \underline{4x^3\sin(x^2)} - \underline{2x\cos(x^2)} + \underline{4x^3\sin(x^2)} = 0$ ∎

b.) Use the method of reduction of order to find the general solution of the DE.

$y = v\sin(x^2)$

$y' = v'\sin(x^2) + 2xv\cos(x^2)$

$y'' = v''\sin(x^2) + 4xv'\cos(x^2) + v(2\cos(x^2) - 4x^2\sin(x^2))$

$xy'' - y + 4x^3 y = v''(x\sin(x^2)) + v'(4x^2\sin(x^2) - \sin(x^2)) + v(0) = 0$

$\to \quad v'' = \dfrac{1-4x^2}{x} v'$

$\int \frac{1}{v'}\, d(v') = \int \left(\frac{1}{x} - 4x\right) dx \quad \to \quad \ln(v') = \ln x - 2x^2 + C_1$

$\to \quad v' = C_1 x e^{-2x^2}$

$v = C_1 \int \underline{-4x}\, e^{-2x^2} \underline{dx} = C_1 e^{-2x^2} + C_2$

So, $\quad y = v\sin(x^2) = (C_1 e^{-2x^2} + C_2)\sin(x^2) = \boxed{C_1 e^{-2x^2}\sin(x^2) + C_2\sin(x^2) = y}$

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