Name: Key

M555: Differential Equations I (Su.19)
Good Problems 5
Sections 5.1 and 5.2
Due: Wednesday, 3 July 2019 at 9:50 am

WICHITA STATE UNIVERSITY

Instructions. Complete all problems on this paper. You may use any resources that you'd like, but be sure to show enough work.

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1. [10 points] Determine the radius of convergence of the power series,

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2 (x+2)^n}{3^n}.$$

RaT: $\left| \frac{\cancel{(-1)^{n+1}}(n+1)^2}{3^{\cancel{n+1}}} \cdot \frac{\cancel{3^n}}{\cancel{(-1)^n}\, n^2} \right| = \frac{1}{3}\left(\frac{n+1}{n}\right)^2 = \frac{1}{3}\left(1+\frac{1}{n}\right)^2 \xrightarrow{n\to\infty} \frac{1}{3} = L$

So the radius of convergence is

$R = 1/L$, or

$\boxed{R=3}$

2. [10 points] use your favorite method to find the Taylor series for the function $f(x) = x^2$ about the point $x_0 = -1$.

$f(x) = x^2 \rightarrow f(-1) = 1$

$f'(x) = 2x \rightarrow f'(-1) = -2$

$f''(x) = 2 \rightarrow f''(-1) = 2$

$\rightarrow x^2 = 1 + (-2)(x-2) + \frac{2}{2!}(x-2)^2$

or

$\boxed{x^2 = 1 - 2(x-2) + (x-2)^2}$

3. [15 points] Use your favorite method to find the Taylor series for $f(x) = \frac{1}{1-x}$ about $x_0 = 0$. What is the interval of convergence?

$$f(x) = \frac{1}{1-x} = \frac{a}{1-r} \quad \text{geometric!}$$

So $\boxed{f(x) = \sum_{n=0}^{\infty} x^n}$

and the interval of convergence is $|r|<1$ or $|x|<1$ or

$$\boxed{-1<x<1}$$

4. [15 points] Use your favorite method to find the Taylor series for $f(x) = \ln(x-1)$ about $x_0 = 2$. What is the interval of convergence?

$$f'(x) = \frac{1}{x-1} = \frac{1}{(x-2)-1+2} = \frac{1}{(x-2)+1} = \frac{1}{1-(-(x-2))} \quad \underline{\text{Geometric}}! \text{ w/ } r = -(x-2)$$

So $f'(x) = \sum_{n=0}^{\infty} (-1)^n (x-2)^n$

thus,

$$f(x) = \ln(x-1) = \int \sum_{n=0}^{\infty} (-1)^n (x-2)^n \, dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (x-2)^{n+1}$$

Plugging in $x=2$ we see that C must equal 0.

So

$$\boxed{\ln(x-1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (x-2)^{n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-2)^n$$

5. [25 points] Find power series solutions to the second order differential equation

$$y'' - xy' - y = 0$$

centered about the point $x_0 = 0$.

$$\left.\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \end{aligned}\right\} \quad y''-xy'-y = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x\sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n$$

$$= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n$$

$$= 2a_2 - a_0 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} - na_n - a_n\right]x^n = 0$$

RR: $2a_2 - a_0 = 0 \rightarrow a_2 = \frac{1}{2}a_0$

$(n+2)(n+1)a_{n+2} - (n+1)a_n = 0 \rightarrow a_{n+2} = \frac{1}{n+2}a_n$ for $n \geq 1$ OR $a_k = \frac{1}{k}a_{k-2}$ for $k \geq 3$,

Table:

k	a_k
0	$a_0 =$ free
1	$a_1 =$ free
2	$a_2 = \frac{1}{2}a_0$
3	$a_3 = \frac{1}{3}a_1$
4	$a_4 = \frac{1}{4}a_2 = \frac{1}{4\cdot 2}a_0$
5	$a_5 = \frac{1}{5}a_3 = \frac{1}{5\cdot 3}a_1$
$\vdots$	
$2n$	$a_{2n} = \frac{1}{(2n)\cdots 1}a_0$ ← evens only $= \frac{1}{2^n(n!)}a_0$
$2n+1$	$a_{2n+1} = \frac{1}{(2n+1)\cdots 1}a_1$ ← odds only $= \frac{2^n(n!)}{(2n+1)!}a_1$

So,

$$\boxed{y = a_0 \sum_{n=0}^{\infty} \frac{1}{2^n(n!)} x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{2^n(n!)}{(2n+1)!} x^{2n+1}}$$

6. [25 points] Find a power series solution to the initial value problem,

$$\begin{cases} xy'' + y' + xy = 0, \\ y(1) = 1, \\ y'(1) = -2. \end{cases}$$

Because of initial condition: center at $x_0 = 1$, $a_0 = 1$, $a_1 = -2$

So, $y = \sum_{n=0}^{\infty} a_n (x-1)^n$

$y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1}$

$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2}$

and $x = (x-1)+1$

$$xy'' + y' + xy = \left[(x-1)+1\right] \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-1} + \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + \left[(x-1)+1\right] \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$= \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-1} + \sum_{n=2}^{\infty} n(n-1) a_n (x-1)^{n-2} + \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} + \sum_{n=0}^{\infty} a_n (x-1)^{n+1} + \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$= \sum_{n=1}^{\infty} (n+1) n a_{n+1} (x-1)^n + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (x-1)^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n + \sum_{n=1}^{\infty} a_{n-1} (x-1)^n + \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$= (2a_2 + a_1 + a_0) + \sum_{n=1}^{\infty} \left[(n+1) n a_{n+1} + (n+2)(n+1) a_{n+2} + (n+1) a_{n+1} + a_{n-1} + a_n\right] (x-1)^n = 0$$

RR: $2a_2 + a_1 + a_0 = 0 \rightarrow a_2 = -\frac{1}{2}(a_0 + a_1) = -\frac{1}{2}(1-2) = \frac{1}{2}$

$(n+2)(n+1) a_{n+2} + (n+1)^2 a_{n+1} + a_n + a_{n-1} = 0$, for $n \geq 1$

$a_{n+2} = \dfrac{-1}{(n+2)(n+1)} \left[(n+1)^2 a_{n+1} + a_n + a_{n-1}\right] \longrightarrow a_k = \dfrac{-1}{k(k-1)} \left[(k-1)^2 a_{k-1} + a_{k-2} + a_{k-3}\right]$ for $k \geq 3$

Table:

k	a_k
0	$a_0 = 1$
1	$a_1 = -2$
2	$a_2 = \frac{1}{2}$
3	$a_3 = \frac{-1}{3\cdot 2}\left[4a_2 + a_1 + a_0\right] = \frac{-1}{6}\left[2-2+1\right] = \frac{-1}{6}$
4	$a_4 = \frac{-1}{4\cdot 3}\left[9a_3 + a_2 + a_1\right] = \frac{-1}{12}\left[-\frac{9}{6} + \frac{1}{2} - 2\right] = \frac{-1}{12}\left[\frac{-9+3-12}{6}\right] = \frac{18}{6\cdot 3\cdot 4} = \frac{1}{4}$
5	$a_5 = \frac{-1}{5\cdot 4}\left[16a_4 + a_3 + a_2\right] = \frac{-1}{20}\left[4 - \frac{1}{6} + \frac{1}{2}\right] = \frac{-1}{20}\left[\frac{24-1+3}{6}\right] = \frac{-26}{20\cdot 6} = \frac{-13}{60}$

So,

$$\boxed{y = 1 - 2(x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3 + \frac{1}{4}(x-1)^4 - \frac{13}{60}(x-1)^5 + \cdots}$$

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