Name: Key

M555: Differential Equations I (Su.19)
Good Problems 6
Sections 5.2 – 5.5

Instructions. Complete all problems, showing enough work. All work must be done on this paper. You may use your own hand-written notes, but you may not use any electronic devices.

1. [25 points] Consider the *Hermite Equation*,

$$y'' - 2xy' + \lambda y = 0,$$

where λ is a real constant. Find and clearly identify the recurrence relation for the solution centered at $x_0 = 0$. You do **not** need to solve for the coefficients.

$$\left.\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n \\ y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n \end{aligned}\right\}$$

$$y'' - 2xy' + \lambda y =$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} - \sum_{n=1}^{\infty} 2n a_n x^n + \sum_{n=0}^{\infty} \lambda a_n x^n$$

$$= (2a_2 + \lambda a_0) + \sum_{n=1}^{\infty} \left[(n+2)(n+1) a_{n+2} - 2n a_n + \lambda a_n\right] x^n = 0$$

RR: $\boxed{a_2 = \frac{-\lambda}{2} a_0 \quad , \quad a_{n+2} = \frac{(2n-\lambda)\, a_n}{(n+2)(n+1)} \quad , \quad n \geq 1}$

OR

$$\boxed{\begin{aligned} a_2 &= \frac{-\lambda}{2} a_0 \\ a_k &= \frac{(2k-4-\lambda)\, a_{k-2}}{k(k-1)} \quad , \quad k \geq 3 \end{aligned}}$$

2. [30 points] Consider the differential equation

$$y'' + (\sin x)\,y = 0.$$

Use your favorite method to find the first three non-zero terms of each of the power series solutions centered at $x_0 = 0$. You do **not** need to write the solution in Σ-notation.

$\varphi^{(n)}(0) = n!\,a_n$

$$\boxed{\begin{array}{l}\varphi(0) = a_0 \\ \varphi'(0) = a_1\end{array}}$$

$\varphi'' = -\sin x \cdot \varphi$

$\varphi''(0) = -\sin 0 \cdot a_0 = 0$, so $\boxed{a_2 = 0}$

$\varphi''' = -\cos x \cdot \varphi - \sin x \cdot \varphi'$

$\varphi'''(0) = -a_0 = 3!\,a_3 \Rightarrow \boxed{a_3 = \frac{-1}{3!} a_0}$

$\varphi^{(4)} = \sin x \cdot \varphi - 2\cos x \cdot \varphi' - \sin x \cdot \varphi''$

$\varphi^{(4)}(0) = -2\varphi'(0) = -2a_1 = 4!\,a_4 \Rightarrow \boxed{a_4 = \frac{-2}{4!} a_1}$

$\varphi^{(5)} = \cos x \cdot \varphi + \sin x \cdot \varphi' + 2\sin x \cdot \varphi' - 2\cos x \cdot \varphi'' - \cos x \cdot \varphi'' - \sin x \cdot \varphi'''$

$= \cos x \cdot \varphi + 3\sin x \cdot \varphi' - 3\cos x \cdot \varphi'' - \sin x \cdot \varphi'''$

$\varphi^{(5)}(0) = \varphi(0) - 3\varphi''(0) = a_0 - 3\cdot 2!\,a_2 = a_0 - 3!\,a_2 = 5!\,a_5 \Rightarrow a_5 = \frac{1}{5!}a_0 - \frac{3!}{5!}a_2 = \frac{1}{5!}a_0 + 0 = \boxed{\frac{1}{5!}a_0 = a_5}$

$\varphi^{(6)} = -\sin x \cdot \varphi + \cos x \cdot \varphi' + 3\cos x \cdot \varphi' + 3\sin x \cdot \varphi'' + 3\sin x\, \varphi'' - 3\cos x \cdot \varphi''' - \cos x \cdot \varphi''' - \sin x \cdot \varphi^{(4)}$

$= -\sin x\,\varphi + 4\cos x \cdot \varphi' + 6\sin x \cdot \varphi'' - 4\cos x \cdot \varphi''' - \sin x \cdot \varphi^{(4)}$

$\varphi^{(6)}(0) = 4\varphi'(0) - 4\varphi'''(0) = 4a_1 - 4\cdot 3!\,a_3 = 4a_1 + 4a_0 = 6!\,a_6 \Rightarrow \boxed{a_6 = \frac{4}{6!}a_0 + \frac{4}{6!}a_1}$

$y_1 = a_0\left(1 - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \frac{4}{6!}x^6 + \cdots\right)$

$y_2 = a_1\left(x - \frac{2}{4!}x^4 + \frac{4}{6!}x^6 + \cdots\right)$

- or it can be done by series methods.

3. [15 points] Find and classify (as regular or irregular) all singular points of the differential equation.

$$x^2(1-x)y'' + (x-2)y' - 3xy = 0.$$

Do **not** solve the equation.

<u>Singular points:</u> $x=0$ and $x=1$.

$x=0$:

$$\lim_{x\to 0} \frac{x\,(x-2)}{x^2(1-x)} = \pm\infty \qquad \boxed{\text{Irregular}}$$

$x=1$:

$$\lim_{x\to 1} \frac{(x-1)\,(x-2)}{x^2\,(1-x)} = \lim_{x\to 1} \frac{2-x}{x^2} = \frac{1}{1^2} = 1$$

$$\lim_{x\to 1} \frac{(x-1)^{\cancel{2}}\,(+3x)}{x^2\,(\cancel{1-x})} = \lim_{x\to 1} \frac{3x^2-3x}{x^2} = 3$$

$$\boxed{\text{Regular}}$$

4. [30 points] Consider the differential equation

$$xy'' + y = 0.$$

Show that $x_0 = 0$ is a regular singular point; find the exponents at $x_0 = 0$; ~~and find the first four non-zero terms of the series solution corresponding to the larger exponent.~~

$p_0 = \lim_{x \to 0} x \frac{0}{x} = 0.$

$q_0 = \lim_{x \to 0} x^2 \frac{1}{x} = 0.$

Indicial Equation: $r^2 + (0-1)r + 0 = 0$

$r^2 - r = 0$

$r(r-1) = 0$

$\boxed{r = 0, 1.}$

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