Name: Key

M555: Differential Equations I (Su.19)

Good Problems 7

Sections 5.5, 6.1–6.3

Instructions. Complete all problems, showing enough work. All work must be done on this paper. You may use your own hand-written notes, but you may not use any electronic devices.

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1. [30 points] Consider the second order differential equation

$$x^2y'' + xy' + (x-2)y = 0.$$

(*a.*) Show that $x_0 = 0$ is a regular singular point; (*b.*) Determine the indicial equation and the exponents at the singularity; and (*c.*) Find the series solution ($x > 0$) corresponding to the larger exponent.

a.) $\left.\begin{aligned} &\lim_{x\to 0} x\cdot\frac{x}{x^2} = 1 \\ &\lim_{x\to 0} x^2\frac{(x-2)}{x^2} = -2 \end{aligned}\right\}$ so $x_0 = 0$ is regular

b.) The indicial equation is $r^2 + (1-1)r - 2 = 0$

$$\boxed{r^2 - 2 = 0}$$

and the exponents are $\boxed{r = \pm\sqrt{2}}$.

c.) $y = \sum_{n=0}^{\infty} a_n x^{n+r}$

$y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$

$y'' = \sum_{n=0}^{\infty} (n+r-1)(n+r)a_n x^{n+r-2}$

$x^2y'' + xy' + (x-2)y =$

$= \sum_{n=0}^{\infty}(n+r-1)(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+1} - \sum_{n=0}^{\infty} 2a_n x^{n+r}$

$= \underline{(r-1)r a_0 x^r + r a_0 x^r - 2a_0 x^r} + \sum_{n=1}^{\infty}\Big[(n+r-1)(n+r)a_n + (n+r)a_n - 2a_n + a_{n-1}\Big]x^{n+r}$

$(r^2 - r + r - 2)a_0 x^r = 0 \Rightarrow r^2 - 2 = 0$ (Indicial Equation from above)

$a_0 =$ free

$\Big[(n+r-1)(n+r) + n+r - 2\Big]a_n = -a_{n-1}$

$\Big[(n+r)^2 - 2\Big]a_n = -a_{n-1}$

RR: $\boxed{a_n = \dfrac{-1}{(n+r)^2 - 2}a_{n-1} \quad \text{for } n \geq 1}$

$r = \sqrt{2}$:

n	a_n
0	a_0
1	$a_1 = \dfrac{-1}{(1+\sqrt{2})^2 - 2}a_0$
2	$a_2 = \dfrac{-1}{(2+\sqrt{2})^2 - 2}a_1 = \dfrac{1}{[(1+\sqrt{2})^2 - 2][(2+\sqrt{2})^2 - 2]}a_0$
3	$a_3 = \dfrac{-1}{(3+\sqrt{2})^2 - 2}a_2 = \dfrac{-1}{[(1+\sqrt{2})^2 - 2][(2+\sqrt{2})^2 - 2][(3+\sqrt{2})^2 - 2]}a_0$

so $\boxed{y(x) = a_0 x^{\sqrt{2}}\left(1 - \frac{1}{(1+\sqrt{2})^2 - 2}x + \frac{1}{[(1+\sqrt{2})^2 - 2][(2+\sqrt{2})^2 - 2]}x^2 - \frac{1}{[(1+\sqrt{2})^2 - 2][(2+\sqrt{2})^2 - 2][(3+\sqrt{2})^2 - 2]}x^3 + \cdots\right)}$

2. [20 points] Use the <u>definition</u> of the Laplace transform to compute

$$\mathscr{L}\{t\sin(2t)\}.$$

You must use the definition to receive credit. Be sure to treat any improper integrals properly.

$$\mathcal{L}\{t\sin(2t)\} = \int_0^\infty t\sin(2t)\,e^{-st}\,dt = \int_0^\infty t\cdot\frac{1}{2i}\left(e^{2it}-e^{-2it}\right)e^{-st}\,dt$$

$$= \frac{1}{2i}\int_0^\infty \underbrace{t}_{u}\,\underbrace{e^{-(s-2i)t}\,dt}_{dv} - \frac{1}{2i}\int_0^\infty \underbrace{t}_{u}\,\underbrace{e^{-(s+2i)t}\,dt}_{dv}$$

	u	dv
$+$	t	$e^{-(s-2i)t}$
$-$	1	$\frac{-1}{(s-2i)}e^{-(s-2i)t}$
	0	$\frac{1}{(s-2i)^2}e^{-(s-2i)t}$

	u	dv
$+$	t	$e^{-(s+2i)t}$
$-$	1	$\frac{-1}{(s+2i)}e^{-(s+2i)t}$
	0	$\frac{1}{(s+2i)^2}e^{-(s+2i)t}$

$$= \frac{1}{2i}\left[-e^{-(s-2i)t}\left(\frac{t}{(s-2i)}+\frac{1}{(s-2i)^2}\right) + e^{-(s+2i)t}\left(\frac{t}{(s+2i)}+\frac{1}{(s+2i)^2}\right)\right]\Bigg|_0^A$$

$$= \frac{1}{2i}\left[-e^{-(s-2i)A}\left(\frac{A}{(s-2i)}+\frac{1}{(s-2i)^2}\right) + e^{-(s+2i)A}\left(\frac{A}{s+2i}+\frac{1}{(s+2i)^2}\right) + \frac{1}{(s-2i)^2} - \frac{1}{(s+2i)^2}\right]$$

$$\xrightarrow{A\to\infty} \quad \frac{1}{2i}\left[0 + 0 + \frac{1}{(s-2i)^2} - \frac{1}{(s+2i)^2}\right] \qquad \text{for } s>0$$

Combining the fractions, $= \dfrac{1}{2i}\cdot\dfrac{(s+2i)^2-(s-2i)^2}{\left[(s-2i)(s+2i)\right]^2} = \dfrac{1}{2i}\,\dfrac{\cancel{s^2}+4is-\cancel{4}-(\cancel{s^2}-4is-\cancel{4})}{(s^2+4)^2}$

$$= \frac{1}{2i}\,\frac{8si}{(s^2+4)^2} = \boxed{\frac{4s}{(s^2+4)^2}}$$

3. [20 points] Find the inverse Laplace transform, $\mathscr{L}^{-1}\{F(s)\}$, where

$$F(s) = \frac{2s+2}{s(s^2+4s+5)}.$$

$$F(s) = \frac{2s+2}{s(s^2+4s+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+4s+5}$$

$$\Rightarrow \quad 2s+2 = A(s^2+4s+5) + (Bs+C)s$$

$s=0$: $\quad 2 = 5A \quad$ so $\quad A = \frac{2}{5}$

$s=i$: $\quad 2i+2 = A(i^2+4i+5) + (Bi+C)i$

$$2+2i = 4A + 4Ai - B + Ci$$

$$4A - B = 2 \rightarrow \frac{8}{5} - B = 2 \rightarrow B = \frac{8}{5} - \frac{10}{5} = \frac{-2}{5}$$

$$4A + C = 2 \rightarrow \frac{8}{5} + C = 2 \rightarrow C = \frac{10}{5} - \frac{8}{5} = \frac{2}{5}$$

So, $$\frac{2s+2}{s(s^2+4s+5)} = \frac{2}{5}\left[\frac{1}{s} - \frac{s-1}{(s+2)^2+1}\right] = \frac{2}{5}\left[\frac{1}{s} - \frac{s+2}{(s+2)^2+1} + 3\frac{1}{(s+2)^2+1}\right]$$

So $$\boxed{\mathcal{L}^{-1}\{F(s)\} = \frac{2}{5} - \frac{2}{5}e^{-2t}\cos t + \frac{6}{5}e^{-2t}\sin t}$$

4. [30 points] Use the method of Laplace transforms to solve the initial value problem

$$\begin{cases} y'' + 16y = \cos(2t), \\ y(0) = 1, \\ y'(0) = 0. \end{cases}$$

$\mathcal{L}:$ $s^2 Y - s\cancel{y(0)}_{1} - \cancel{y'(0)} + 16Y = \dfrac{s}{s^2+4}$

$(s^2+16)\,Y = \dfrac{s}{s^2+4} + 1$

$Y = \underbrace{\dfrac{s}{(s^2+4)(s^2+16)}}_{\text{PFD}} + \dfrac{1}{s^2+16}$

$Y = \dfrac{1}{12}\left[\dfrac{s}{s^2+4} - \dfrac{s}{s^2+16}\right] + \dfrac{1}{s^2+16}$

$\dfrac{s}{(s^2+4)(s^2+16)} = \dfrac{As+B}{s^2+4} + \dfrac{Cs+D}{s^2+16}$

$s = (As+B)(s^2+16) + (Cs+D)(s^2+4)$

$s=4i:$ $4i = (4Ci + D)(-16+4)$

$4i = -48Ci - 12D \rightarrow \begin{cases} D = 0 \\ C = \frac{-4}{48} = \frac{-1}{12} \end{cases}$

$s=2i:$ $2i = (2iA + B)(-4+16)$

$2i = 24iA + 24B \rightarrow \begin{cases} B = 0 \\ A = \frac{+1}{12} \end{cases}$

So $\boxed{y(t) = \frac{1}{12}\cos(2t) - \frac{1}{12}\cos(4t) + \sin(4t)}$

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