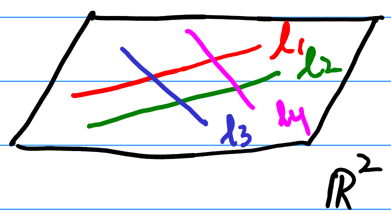
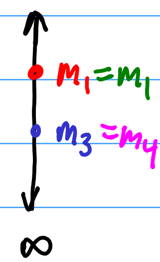
M621 9/17/18

Ch2 - Euclid: A Modern Perspective

§2.1 - Extending the Euclidean Plane



ℓ : Ideal line



Def'n. An ideal point is a point at ∞ where 2 lines w/ a common slope (i.e, parallel lines) intersect.

Thus, there is an ideal point at ∞ corresponding to every possible slope (including lines w/ undefined slope)

Thm 2.1.1. In the Extended plane, any two distinct lines ℓ_1 and ℓ_2 intersect each other in exactly one point.

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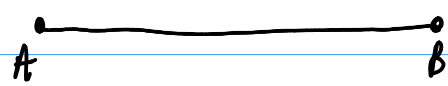
§2.2 - Sensed Magnitudes



AB = the line segment

$\overline{AB}$ = distance from A to B.

$\overline{BA}$ = distance from B to A

$\overline{BA} = -\overline{AB}$

Consider, $\overline{AB} + \overline{BA} = \overline{AB} + (-\overline{AB}) = \overline{AB} - \overline{AB} = 0.$

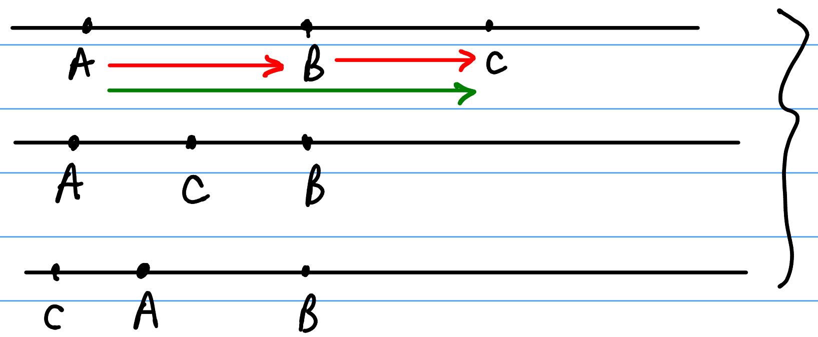
Akin to displacement in physics.

$\overline{AB} + \overline{BA} = \overline{AA} = 0$.

<u>Thm. 2.2.1</u> If A, B, C are collinear pts, then

$$\overline{AB} + \overline{BC} + \overline{CA} = 0$$

<u>"Proof"</u>

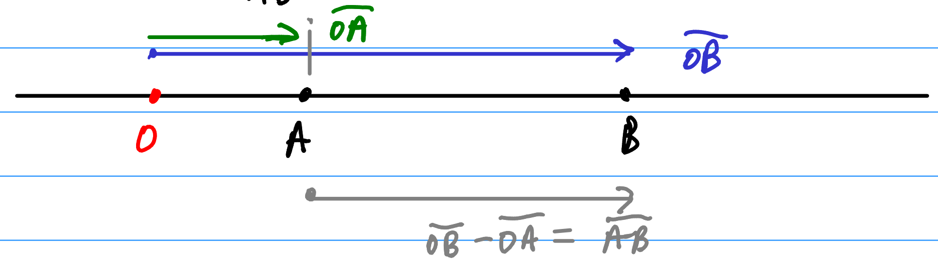


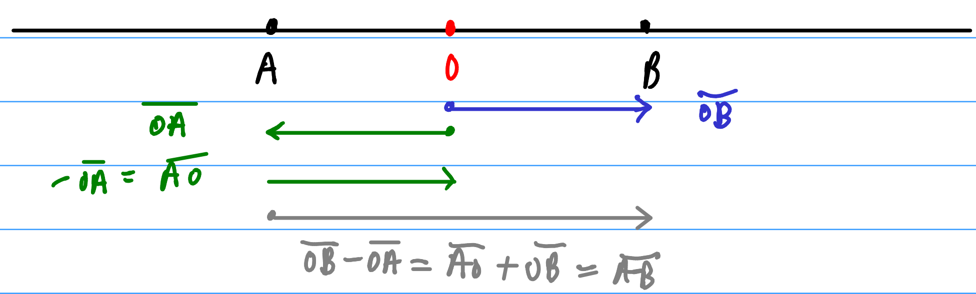
$\overline{AB} + \overline{BC} = -\overline{CA}$

<u>Thm. 2.2.2</u>. Let O be any point on a line AB. Then

$$\overline{AB} = \overline{OB} - \overline{OA}$$

<u>Picture</u>:





We call the point O an origin of $l = AB$.

<u>Thm. 2.2.3</u> (Euler's Theorem) If A, B, C, D are collinear, then

$$\overline{AD} \cdot \overline{BC} + \overline{BD} \cdot \overline{CA} + \overline{CD} \cdot \overline{AB} = 0$$

Proof. use D as an origin for
$$\overline{BC} = \overline{DC} - \overline{DB}$$
$$\overline{CA} = \overline{DA} - \overline{DC}$$
$$\overline{AB} = \overline{DB} - \overline{DA}$$

Plugging in, $\overline{AD}(\overline{DC} - \overline{DB}) + \overline{BD}(\overline{DA} - \overline{DC}) + \overline{CD}(\overline{DB} - \overline{DA})$

$$= \overline{AD}\cdot\overline{DC} - \overline{AD}\cdot\overline{DB} + \overline{BD}\cdot\overline{DA} - \overline{BD}\cdot\overline{DC} + \overline{CD}\cdot\overline{DB} - \overline{CD}\cdot\overline{DA}$$

$$\overline{AD}\cdot\overline{DC} = \overline{DA}\cdot\overline{CD} = \overline{CD}\cdot\overline{DA}$$

$$= 0.$$

Thm. 2.2.4 (Stewart's Theorem) If A, B, C are collinear. For any point P,

[Matthew Stewart Scottish 1746]

$$\overline{PA}^2\cdot\overline{BC} + \overline{PB}^2\cdot\overline{CA} + \overline{PC}^2\cdot\overline{AB} + \overline{BC}\cdot\overline{CA}\cdot\overline{AB} = 0$$

Proof. Two cases: 1.) P is in fact collinear w/ the others.
2.) P is not collinear.

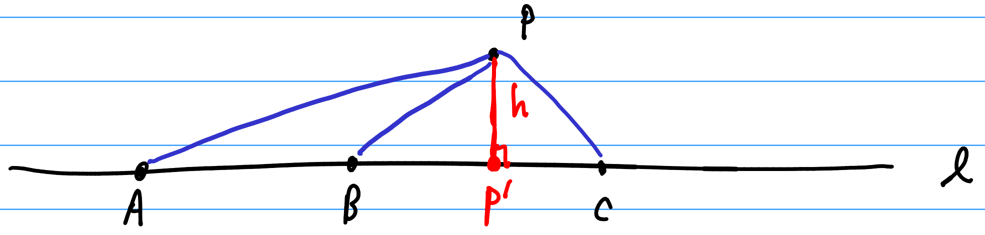
1.) Make P an origin, so that
$$\overline{BC} = \overline{PC} - \overline{PB}$$
$$\overline{CA} = \overline{PA} - \overline{PC}$$
$$\overline{AB} = \overline{PB} - \overline{PA}$$

Plugging in,
$$\overline{PA}^2\cdot\overline{BC} + \overline{PB}^2\cdot\overline{CA} + \overline{PC}^2\cdot\overline{AB} + \overline{BC}\cdot\overline{CA}\cdot\overline{AB}$$
$$= \overline{PA}^2(\overline{PC} - \overline{PB}) + \overline{PB}^2(\overline{PA} - \overline{PC}) + \overline{PC}^2(\overline{PB} - \overline{PA}) + (\overline{PC} - \overline{PB})(\overline{PA} - \overline{PC})(\overline{PB} - \overline{PA})$$

= ... multiply it all out and combine like terms ---

$= 0$

Case 2.)



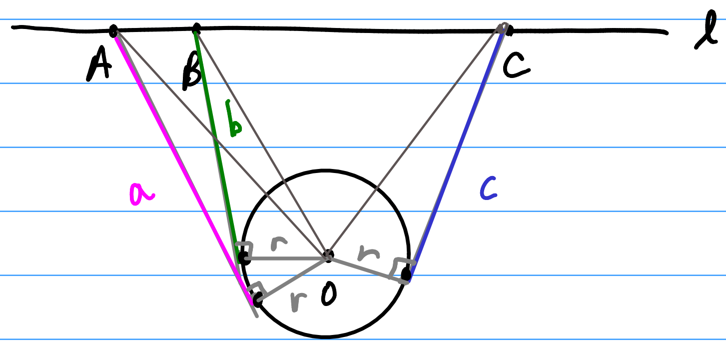
$$\overline{PA}^2 = \overline{P'A}^2 + h^2 \qquad \overline{PC}^2 = \overline{P'C}^2 + h^2$$
$$\overline{PB}^2 = \overline{P'B}^2 + h^2$$

$\overline{PA}^2\,\overline{BC} + \overline{PB}^2\cdot\overline{CA} + \overline{PC}^2\cdot\overline{AB} + \overline{BC}\cdot\overline{CA}\cdot\overline{AB}$

$\underbrace{\overline{P'A}^2\,\overline{BC} + \overline{P'B}^2\cdot\overline{CA} + \overline{P'C}^2\cdot\overline{AB}}_{=0 \text{ by case 1.)}} + h^2\underbrace{(\overline{BC} + \overline{CA} + \overline{AB})}_{=0} + \underbrace{\overline{BC}\cdot\overline{CA}\cdot\overline{AB}}$

$= 0.$

Ex. 2.2.1 A, B, C collinear, a, b, c the tangents from A, B, C to a given circle.



Then,

$$a^2\,\overline{BC} + b^2\,\overline{CA} + c^2\,\overline{AB} + \overline{BC}\cdot\overline{CA}\cdot\overline{AB} = 0$$

Proof. Use Stewart w/ $P = O$.

Apply Pythag: $\overline{OA}^2 = a^2 + r^2$

$\overline{OB}^2 = b^2 + r^2$

$\overline{OC}^2 = c^2 + r^2$

Then rearrange to get $r^2(\overline{BC} + \overline{CA} + \overline{AB}) = 0.$