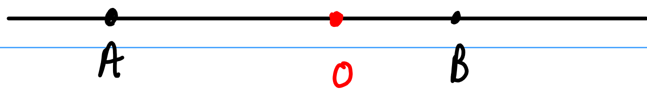
M621 9/19/18

§2.2 Sensed Magnitudes



$\overline{AB} = \overline{OB} - \overline{OA}$ origin

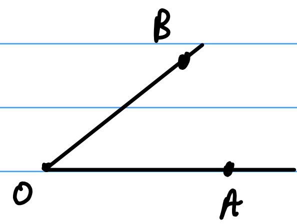
AB

$\overline{AB}$: $A \to B$

$\overline{BA}$: $B \to A$

$\overline{AB} = -\overline{BA}$

Defn 2.2.1



$\measuredangle AOB$ non-directed

$\measuredangle \overline{AOB}$: $OA \to OB$

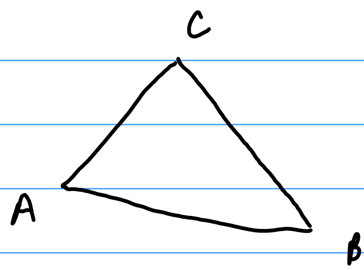
$\measuredangle \overline{BOA}$: $OB \to OA$

$\measuredangle \overline{BOA} = -\measuredangle \overline{AOB}$

Directed angles that open counter-clockwise are positive.

" " " " clockwise " negative.

Defn 2.2.2 –



ΔABC : counter-clockwise

ΔACB : clockwise

$\Delta\overline{ABC}$ = Area of ΔABC

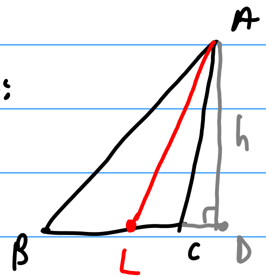
= positive (counter-clockwise)

$\Delta\overline{ACB}$ = negative area (clockwise)

Thm 2.2.5. If a vertex A of triangle $\triangle ABC$ is joined to any point L on BC, then

$$\frac{\overline{BL}}{\overline{LC}} = \frac{AB \sin(\overline{BAL})}{AC \sin(\overline{LAC})}$$

Proof. One case:



Use right triangle trig.

Area $= \frac{1}{2} bc \sin(A)$

$$\frac{\overline{BL}}{\overline{LC}} = \frac{h\,\overline{BL}}{h\,\overline{LC}} = \frac{2\,\Delta \overline{BAL}}{2\,\Delta \overline{LAC}} = \frac{\cancel{2}\left(\cancel{\tfrac{1}{2}}\, \overline{BA}\cdot \cancel{\overline{AL}} \sin(\overline{BAL})\right)}{\cancel{2}\left(\cancel{\tfrac{1}{2}}\, \cancel{\overline{LA}}\cdot \overline{AC} \sin(\overline{LAC})\right)} = \frac{AB \sin(\overline{BAL})}{AC \sin(\overline{LAC})}$$

RE. Can we prove $\dfrac{\overline{BL}}{\overline{LC}} = \dfrac{\overline{AB} \sin(\overline{BAL})}{\overline{AC} \sin(\overline{LAC})}$? Probably not?

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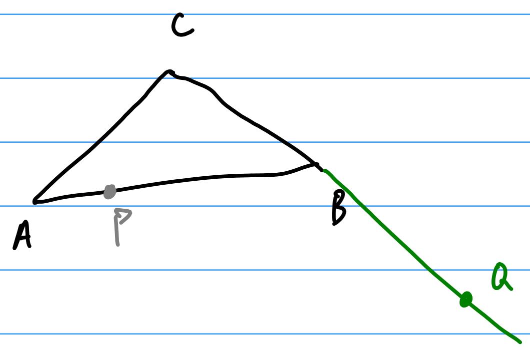
§2.3 Menelaus's and Ceva's Theorems

Defn. 2.3.1. A Menelaus point of an ordinary triangle is a point that lies on an extended side of the triangle but is not a vertex.

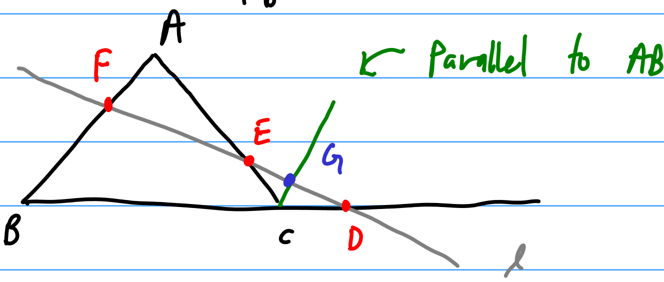


Menelaus's Theorem (Thm. 2.3.1)

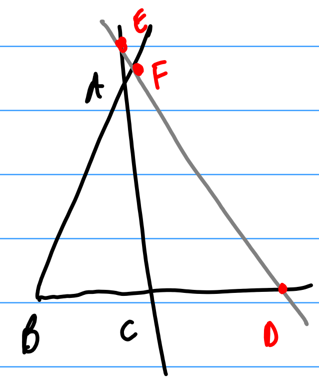
Let $\triangle ABC$ be an ordinary triangle w/ Menelaus points D, E, F on the sides BC, CA, and AB, respectively.
The Menelaus points D, E, F are collinear if and only if

$$\frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} \cdot \frac{\overline{AF}}{\overline{FB}} = -1.$$

Picture:



or



Proof. ($\Rightarrow$, "only if") Suppose D, E, F are collinear.
Draw CG parallel to AB, where G lies on FD.
We get similar triangles: $\triangle FDB \sim \triangle GDC$
$\triangle AEF \sim \triangle CEG$

$\triangle FDB \sim \triangle GDC$: $\quad \dfrac{\overline{FB}}{\overline{GC}} = \dfrac{\overline{BD}}{\overline{CD}} \Rightarrow \dfrac{1}{\overline{GC}} = \dfrac{\overline{BD}}{\overline{FB}\cdot\overline{CD}}$

$\triangle AEF \sim \triangle CEG$: $\quad \dfrac{\overline{CG}}{\overline{AF}} = \dfrac{\overline{CE}}{\overline{AE}} \Rightarrow \overline{CG} = \dfrac{\overline{AF}\cdot\overline{CE}}{\overline{AE}}$

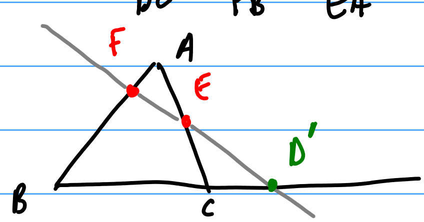
Now, multiply.

$$\frac{\overline{CG}}{\overline{GC}} = \frac{\overline{BD}\cdot\overline{AF}\cdot\overline{CE}}{\overline{CD}\cdot\overline{FB}\cdot\overline{AE}}$$

$$\boxed{-1 = \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{CE}}{\overline{EA}}}$$

☺

($\Leftarrow$, "if") Assume $\dfrac{\overline{BD}}{\overline{DC}} \cdot \dfrac{\overline{AF}}{\overline{FB}} \cdot \dfrac{\overline{CE}}{\overline{EA}} = -1$. TBS: D, E, F are collinear.



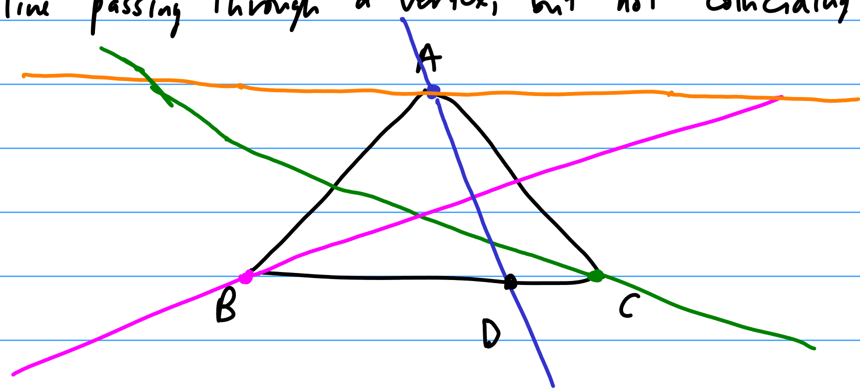
Now, the "only if" half applies to F, E, D'. So

$$\frac{\overline{BD'}}{\overline{D'C}} \cdot \frac{\overline{CE}}{\overline{EA}} \cdot \frac{\overline{AF}}{\overline{FB}} = -1 \underset{\text{by assumption}}{\uparrow}{=} \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} \cdot \frac{\overline{AF}}{\overline{FB}}$$

Then $\frac{\overline{BD'}}{\overline{D'C}} = \frac{\overline{BD}}{\overline{DC}}$ after cancellations.

Then $D' = D$, and D, E, F are in fact collinear. ∎

<u>Def'n 2.3.2.</u> Let ABC be an ordinary triangle. A <u>cevian</u> <u>line</u> is a line passing through a vertex, but not coinciding w/ a side.



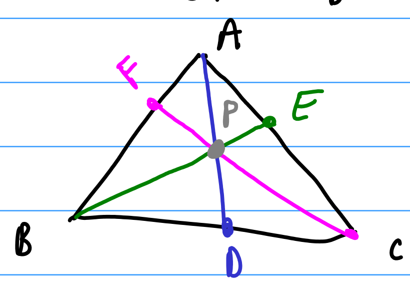
<u>Ceva's Theorem</u> (2.3.2)

Let ΔABC be an ordinary triangle w/ Cevian lines AD, BE, and CF.

The Cevian lines are <u>concurrent</u> if and only if

$$\frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} \cdot \frac{\overline{AF}}{\overline{FB}} = 1.$$

<u>Picture:</u>



ΔADC: B, P, E

$$\boxed{\frac{\overline{AE}}{\overline{EC}} \cdot \frac{\overline{CB}}{\overline{BD}} \cdot \frac{\overline{DP}}{\overline{PA}} = -1} \quad (**)$$

Proof: Menelaus's Thm applies to ΔABD w/ M. pts. C, P, F, so that

$$\boxed{\frac{\overline{BC}}{\overline{CD}} \cdot \frac{\overline{DP}}{\overline{PA}} \cdot \frac{\overline{AF}}{\overline{FB}} = -1} \quad *$$

Similarly for ΔBAE w/ M. pts. C, P, F :

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BP}}{\overline{PE}} \cdot \frac{\overline{EC}}{\overline{CA}} = -1$$

Combining $\frac{(*)}{(**)}$: $1 = \frac{\overline{BC}}{\overline{CD}} \cdot \cancel{\frac{\overline{DP}}{\overline{PA}}} \cdot \frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{EC}}{\overline{AE}} \cdot \frac{\overline{BD}}{\overline{CB}} \cdot \cancel{\frac{\overline{PA}}{\overline{DP}}}$

$$\boxed{\frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} \cdot \frac{\overline{AF}}{\overline{FB}} = 1}$$