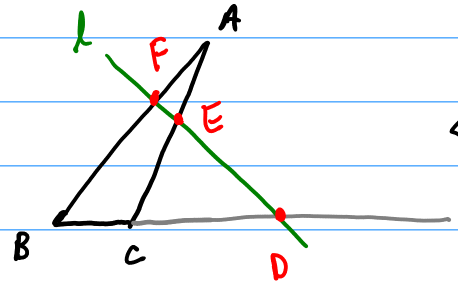
M621 9/24/18

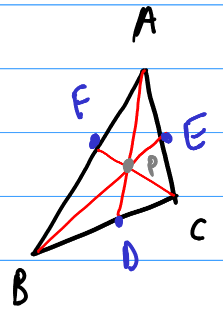
§2.3 – Menelaus's and Ceva's Theorems

ΔABC:
Menelaus



$\Leftrightarrow \quad \dfrac{\overline{AF}}{\overline{FB}} \cdot \dfrac{\overline{BD}}{\overline{DC}} \cdot \dfrac{\overline{CE}}{\overline{EA}} = -1$

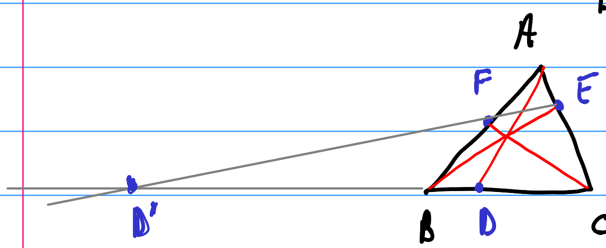
ΔABC:
Ceva



$\Leftrightarrow \quad \dfrac{\overline{AF}}{\overline{FB}} \cdot \dfrac{\overline{BD}}{\overline{DC}} \cdot \dfrac{\overline{CE}}{\overline{EA}} = 1$

Thm. 2.3.5. ΔABC be an ordinary triangle. If AD, BE, and CF are concurrent Cevian lines, and if D' denotes the intersection of FE w/ side BC, then

$$\frac{\overline{BD}}{\overline{DC}} = -\frac{\overline{BD'}}{\overline{D'C}}.$$



F, E, D' are collinear Menelaus pts.

Ceva's Thm: $1 = \boxed{\dfrac{\overline{AF}}{\overline{FB}}} \cdot \dfrac{\overline{BD}}{\overline{DC}} \cdot \boxed{\dfrac{\overline{CE}}{\overline{EA}}}$

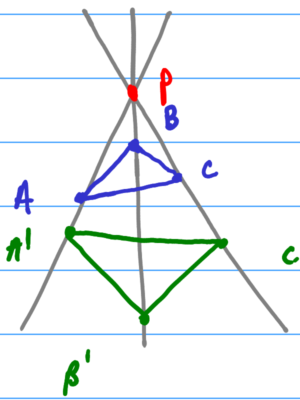
Menelaus: $-1 = \boxed{\dfrac{\overline{AF}}{\overline{FB}}} \cdot \dfrac{\overline{BD'}}{\overline{D'C}} \cdot \boxed{\dfrac{\overline{CE}}{\overline{EA}}}$

$\Rightarrow \quad \dfrac{\overline{BD}}{\overline{DC}} = -\dfrac{\overline{BD'}}{\overline{D'C}} \quad \blacksquare$

Construction. 2.3.1 – Given a line segment AB w/ D. Construct the pt D' satisfying the theorem.

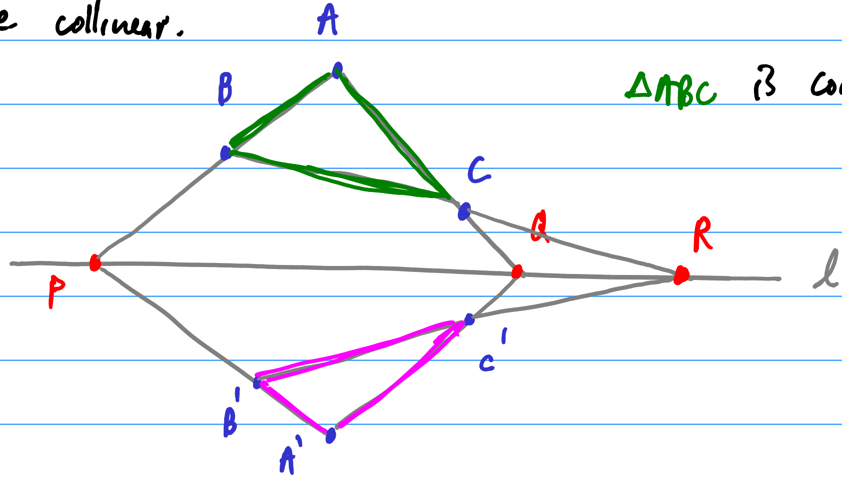
– Constructed on board.

Defn. Two triangles ΔABC and $\Delta A'B'C'$ are said to be copolar if and only if AA', BB', and CC' are concurrent.



ΔABC is copolar to $\Delta A'B'C'$

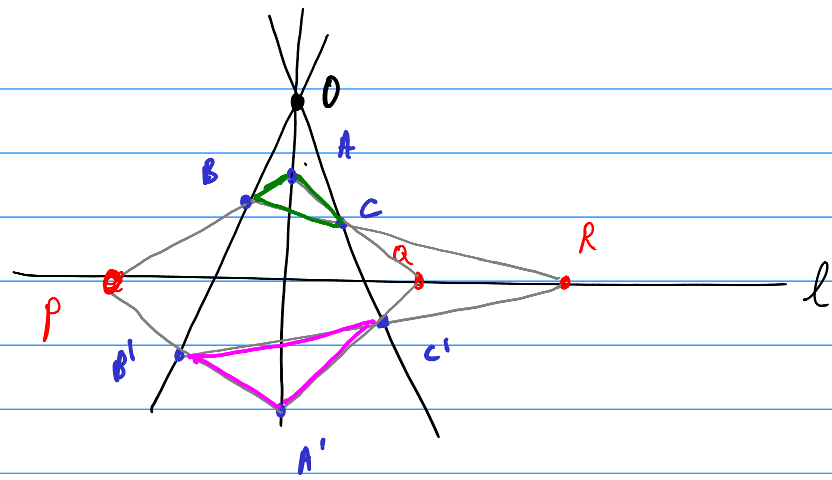
Defn. Two triangles ΔABC and $\Delta A'B'C'$ are said to be coaxial iff the intersection points of AB w/ $A'B'$, BC w/ $B'C'$, and AC w/ $A'C'$ are collinear.



ΔABC is coaxial w/ $\Delta A'B'C'$

Thm. (Desargues) Copolar triangles are coaxial, and conversely.

"Proof":



Assume ΔABC and $\Delta A'B'C'$ are copolar at O.

ΔBCO has Menelaus pts B', C', R. Therefore

$$\frac{\overline{OB'}}{\overline{B'B}} \cdot \frac{\overline{BR}}{\overline{RC}} \cdot \frac{\overline{CC'}}{\overline{C'O}} = -1$$

Similarly for ΔACO and ΔABO we have

$$\frac{\overline{OA'}}{\overline{A'O}} \cdot \frac{\overline{AQ}}{\overline{QC}} \cdot \frac{\overline{CC'}}{\overline{C'O}} = -1 \quad \text{and} \quad \frac{\overline{OA'}}{\overline{A'A}} \cdot \frac{\overline{AP}}{\overline{PB}} \cdot \frac{\overline{BB'}}{\overline{B'O}} = -1$$

Combining all of these (carefully), we obtain:

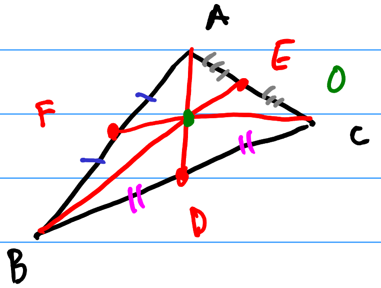
$$\frac{\overline{AP}}{\overline{PB}} \cdot \frac{\overline{BR}}{\overline{RC}} \cdot \frac{\overline{CQ}}{\overline{QA}} = -1$$

By Menelaus's Theorem for ΔABC w/ pt P, Q, R, $P, Q,$ and R must be collinear and the triangles are coaxial.

RE. Finish the other direction. ∎

Examples. (Exercises.)

1. Medians of any triangle are concurrent.
This point is the centroid of $\triangle ABC$.



Then, $\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1 \cdot 1 \cdot 1 = 1.$

By Ceva's Theorem the segments AD, BE, and CF are concurrent. ∎

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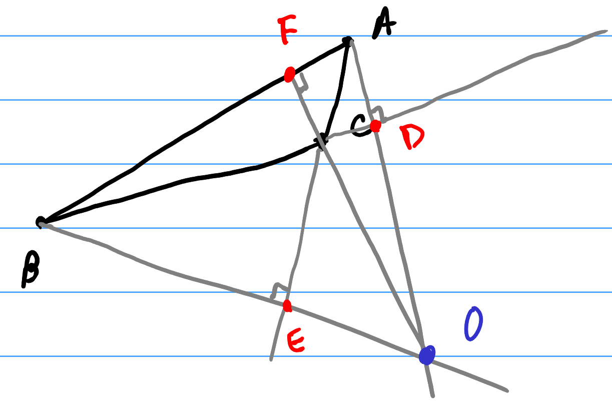
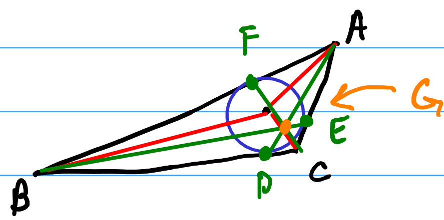
2. The altitudes of $\triangle ABC$ are concurrent : orthocenter.



use the trig. form of Ceva's Theorem.

4. <u>Gergonne Pt</u>. Let $O(r)$ be the inscribed circle of ΔABC. Let D, E, F be the points of tangency. Then $AD, BE,$ and CF concurrent at the <u>Gergonne</u> pt.



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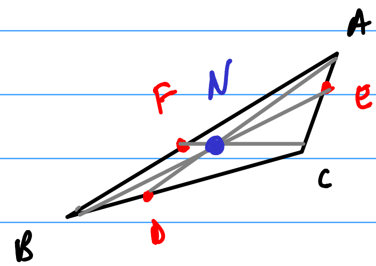
5. <u>Nagel Pt</u>: ΔABC. Let D, E, F be the points "halfway around" the triangle from each vertex:



each is $= S$. $\left\{ \begin{array}{l} \overline{AB} + \overline{BD} = \overline{DC} + \overline{CA} \\ \overline{BC} + \overline{CE} = \overline{EA} + \overline{AB} \\ \overline{CA} + \overline{AF} = \overline{FB} + \overline{BC} \end{array} \right.$