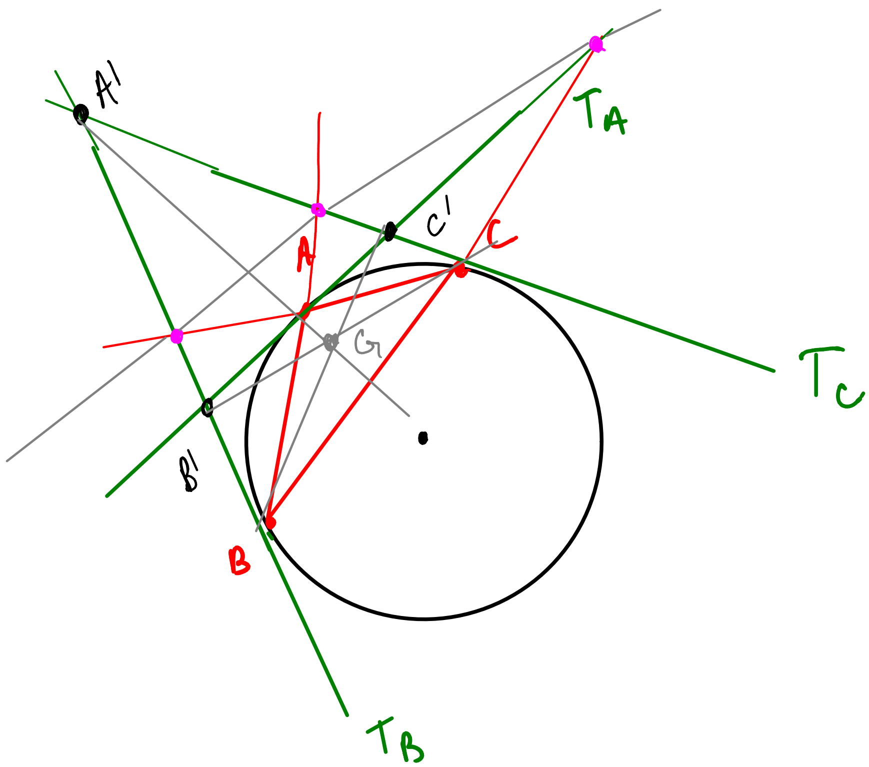
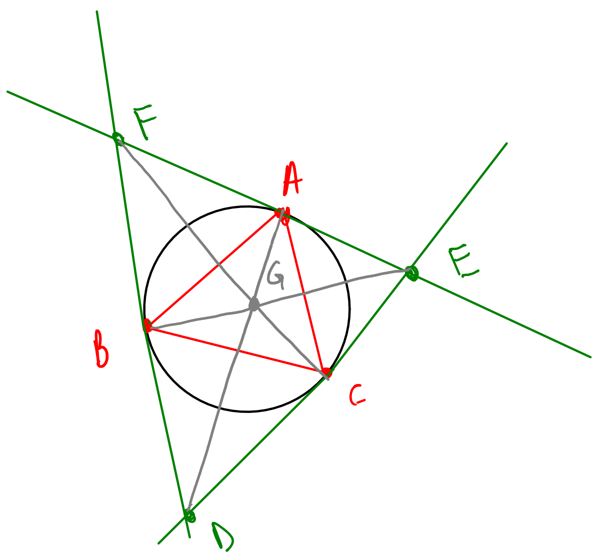
M621 10/1/18

GP 13. 2.3.10



Try again:



G is the Gergonne pt of Δ DEF:
CF, AD, BE are concurrent at G.
Therefore, ΔDEF and ΔABC are copolar.
By Desargues Thm, ΔDEF and ΔABC are coaxial:
the intersection pts of "common" sides are collinear. $\square$

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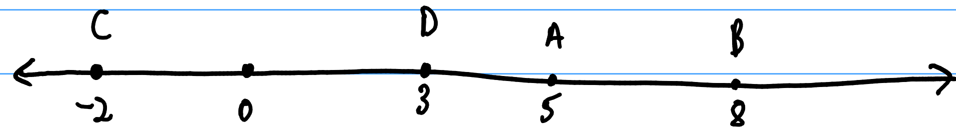
§2.5 Cross Ratio & §2.6 Harmonic Division

Def'n. Let A, B, C, D be collinear points, all distinct. The cross ratio is

$$(AB, CD) = \left(\frac{\overline{AC}}{\overline{CB}}\right) \Big/ \left(\frac{\overline{AD}}{\overline{DB}}\right) = \frac{\overline{AC}}{\overline{CB}} \cdot \frac{\overline{DB}}{\overline{AD}}.$$

Ex.



$$(AB, CD) = \frac{\overline{AC}}{\overline{CB}} \Big/ \frac{\overline{AD}}{\overline{DB}} = \frac{-7}{10} \Big/ \frac{-2}{5} = \frac{7}{10} \cdot \frac{5}{2} = \frac{7}{4}.$$

$$(CA, DB) = \frac{\overline{CD}}{\overline{DA}} \Big/ \frac{\overline{CB}}{\overline{BA}} = \frac{5}{2} \Big/ \frac{10}{-3} = -\frac{5}{2} \cdot \frac{3}{10} = -\frac{3}{4}.$$

Thm. 2.5.1 If $(AB, CD) = r$ and in the symbol (AB, CD) we:

1. interchange any two points and also interchange the other two points, then the value is unchanged.
   e.g. $(AB, CD) = (CD, AB) = (BA, DC)$

2. interchange the first pair of points only, then $(BA, CD) = \frac{1}{r}$

3. interchange the middle point, then $(AC, BD) = 1 - r$.

Proof. (part 3.) $(AB, CD) = r = \frac{\overline{AC}}{\overline{CB}} \cdot \frac{\overline{DB}}{\overline{AD}} = r.$

Consider $(AC, BD) = \frac{\overline{AB}}{\overline{BC}} \cdot \frac{\overline{DC}}{\overline{AD}}$ use C as an origin for $\overline{AB}$ and B for $\overline{DC}$.

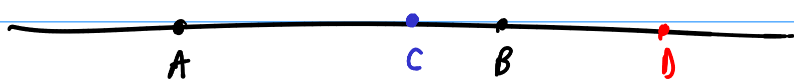
Then $(AC,BD) = \dfrac{(\overline{AC}+\overline{CB})\cdot(\overline{DB}+\overline{BC})}{-\overline{CB}\cdot\overline{AD}}$

$$= \frac{\overline{AC}\cdot\overline{DB} + (\overline{AC}\cdot\overline{BC}+\overline{CB}\cdot\overline{DB}+\overline{CB}\cdot\overline{BC})}{-\overline{CB}\cdot\overline{AD}}$$

$$= \underbrace{-\frac{\overline{AC}\cdot\overline{DB}}{\overline{CB}\cdot\overline{AD}}}_{-r} + \underbrace{\frac{\cancel{\overline{BC}}\,\big(\overbrace{\boxed{\overline{AC}+\overline{BD}+\overline{CB}}}^{\overline{AD}}\big)}{\cancel{-\overline{CB}}\cdot\overline{AD}}}_{1}$$

$$= 1-r \quad \blacksquare$$

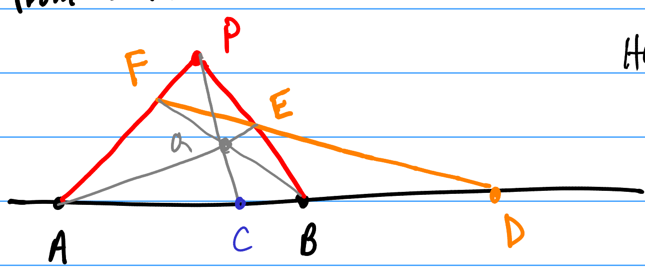
<u>Defn. 2.6.1.</u> Let A, B, C, D be distinct collinear points such that $(AB,CD) = -1$.
Then we say AB is <u>divided harmonically</u> by C and D.
C and D are said to be <u>harmonic conjugates</u> wrt the line segment AB.



$$(AB,CD) = -1 = \frac{\overline{AC}}{\overline{CB}} \Big/ \frac{\overline{AD}}{\overline{DB}} \Rightarrow \frac{\overline{AC}}{\overline{CB}} = -\frac{\overline{AD}}{\overline{DB}}.$$

C and D divide $\overline{AB}$ in the same ratio: one inside and one outside.

Recall a thm from §2.3:



Here $\dfrac{\overline{AC}}{\overline{CB}} = -\dfrac{\overline{AD}}{\overline{DB}}$!

This is one way to construct one harmonic conjugate, given the other.

Two other methods — on board: