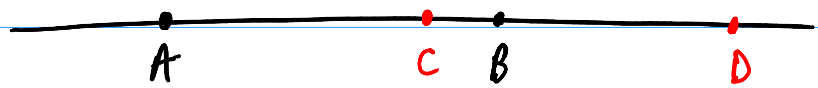
M621 10/3/18

Midterm is on W, 17 Oct 2018.

Recall. If A, B, C, D are collinear, then define the cross ratio to be

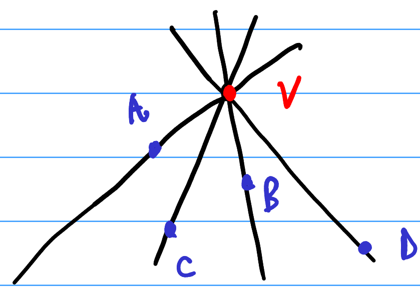
$$(AB, CD) = \left(\frac{\overline{AC}}{\overline{CB}}\right) \Big/ \left(\frac{\overline{AD}}{\overline{DB}}\right) = \frac{\overline{AC}}{\overline{CB}} \cdot \frac{\overline{DB}}{\overline{AD}}$$

If $(AB, CD) = -1$ then C, D are harmonic conjugates wrt $\overline{AB}$.



§2.5

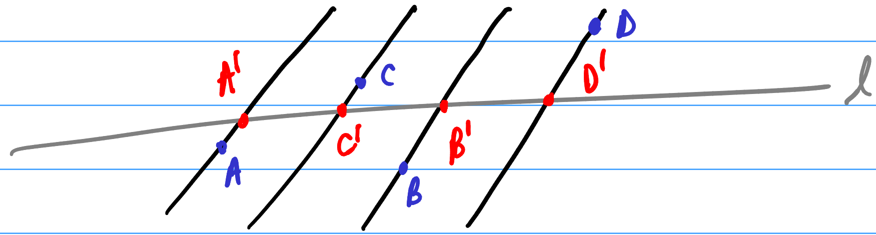
Def'n. 2.5.2 Let VA, VB, VC, VD be a pencil of lines



The cross ratio of the pencil is:

$$V(AB, CD) = \left(\frac{\sin \widehat{AVC}}{\sin \widehat{CVB}}\right) \Big/ \left(\frac{\sin \widehat{AVD}}{\sin \widehat{DVB}}\right)$$

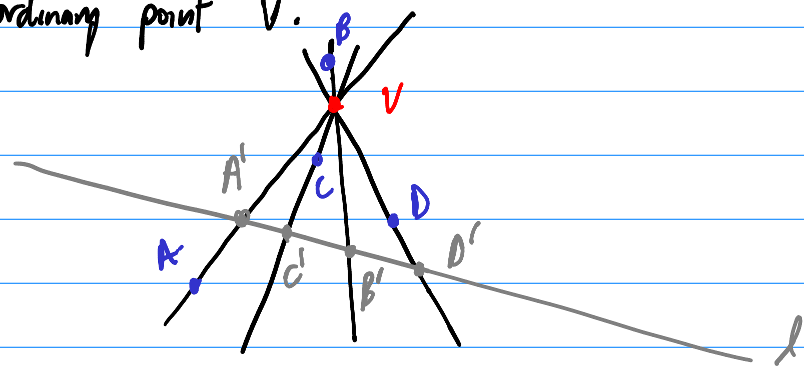
Thm 2.5.3. If V is an ideal point, then VA, VB, VC, VD are parallel.



Then $(A'B', C'D')$ does not depend on the choice of line (transversal) l. We define

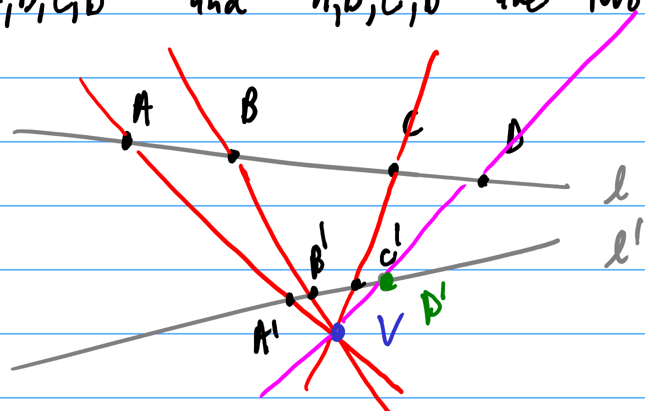
$$V(AB, CD) = (A'B', C'D').$$

Thm 2.5.4. Let VA, VB, VC, VD be a pencil of lines concurrent at an ordinary point V.



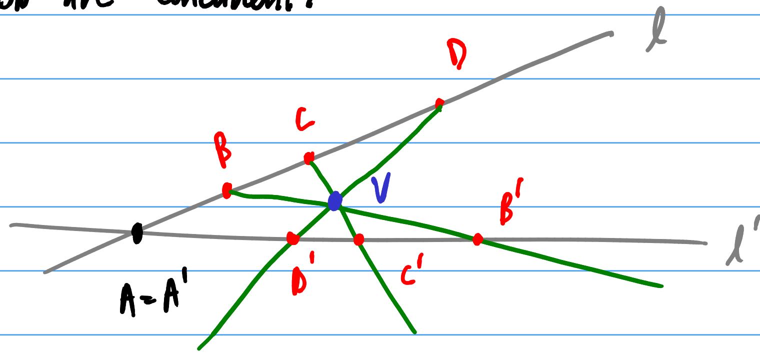
Then, $V(AB, CD) = (A'B', C'D')$ for any transversal l.

Cor 2.5.1 A, B, C, D and A', B', C', D' are two set collinear pts.

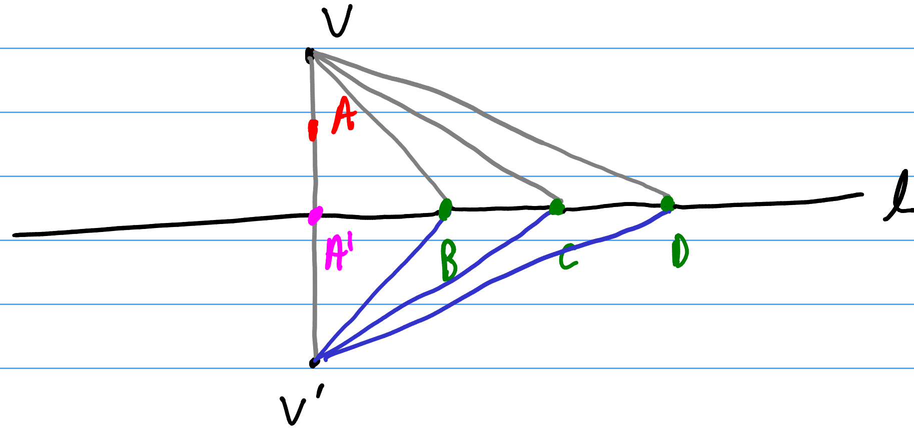


If $(AB, CD) = (A'B', C'D')$, and if AA', BB', and CC' are concurrent, then DD' is also concurrent.

Cor 2.5.2. A, B, C, D collinear and A', B', C', D' collinear such that $(AB, CD) = (A'B', C'D')$. If $A = A'$, then BB', CC', and DD' are concurrent.



Cor. 2.5.4 Let VA, VB, VC, VD be a pencil lines, and $V'A, V'B, V'C, V'D$ another pencil. Suppose A lies on VV'. Then $V(AB,CD) = V'(AB,CD)$ if and only if B, C, D are collinear.



Thm. 2.6.3 $\underbrace{(AB,CD) = -1}$ if and only if $\frac{2}{\overline{AB}} = \frac{1}{\overline{AC}} + \frac{1}{\overline{AD}}$

Proof. $\frac{\overline{AC}}{\overline{CB}} \cdot \frac{\overline{DB}}{\overline{AD}} = -1$ implying $\frac{\overline{AC}}{\overline{CB}} = -\frac{\overline{AD}}{\overline{DB}}$

or $\frac{\overline{CB}}{\overline{AC}} = -\frac{\overline{DB}}{\overline{AD}}$.

Then divide both terms by $\overline{AB}$,

$$\frac{\overline{CB}}{\overline{AC}\cdot\overline{AB}} = \frac{-\overline{DB}}{\overline{AD}\cdot\overline{AB}}$$ and use A as an origin for $\overline{CB}$ and $\overline{DB}$

$$\frac{\overline{AB}-\overline{AC}}{\overline{AC}\cdot\overline{AB}} = \frac{-(\overline{AB}-\overline{AD})}{\overline{AD}\cdot\overline{AB}}$$

$$\frac{\overset{1}{\cancel{\overline{AB}}}}{\overline{AC}\cdot\cancel{\overline{AB}}} - \frac{\overset{1}{\cancel{\overline{AC}}}}{\cancel{\overline{AC}}\cdot\overline{AB}} + \frac{\overset{1}{\cancel{\overline{AB}}}}{\overline{AD}\cdot\cancel{\overline{AB}}} - \frac{\overset{1}{\cancel{\overline{AD}}}}{\cancel{\overline{AD}}\cdot\overline{AB}} = 0$$

Then combine like terms and rearrange to get

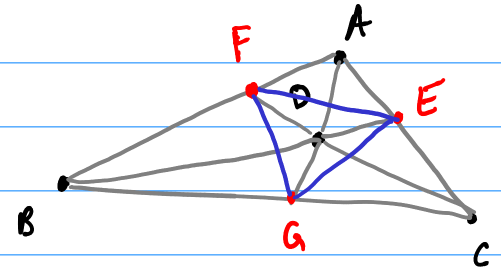
$$\frac{1}{\overline{AC}} + \frac{1}{\overline{AD}} = \frac{2}{\overline{AB}} \quad \blacksquare$$

Thm. 2.6.4 $(AB, CD) = -1$ iff $\underbrace{\overline{OB}^2 = \overline{OC}\cdot\overline{OD}}$ where O is the midpoint of AB.

Proof: $(AB, CD) = -1 \Rightarrow \frac{\overline{AC}}{\overline{CB}} = -\frac{\overline{AD}}{\overline{DB}}$

Use O an origin for everything, cross multiply, combine & cancel.
Keep in mind $\overline{AO} = \overline{OB}$ and $\overline{OA} = \overline{BO}$. ∎

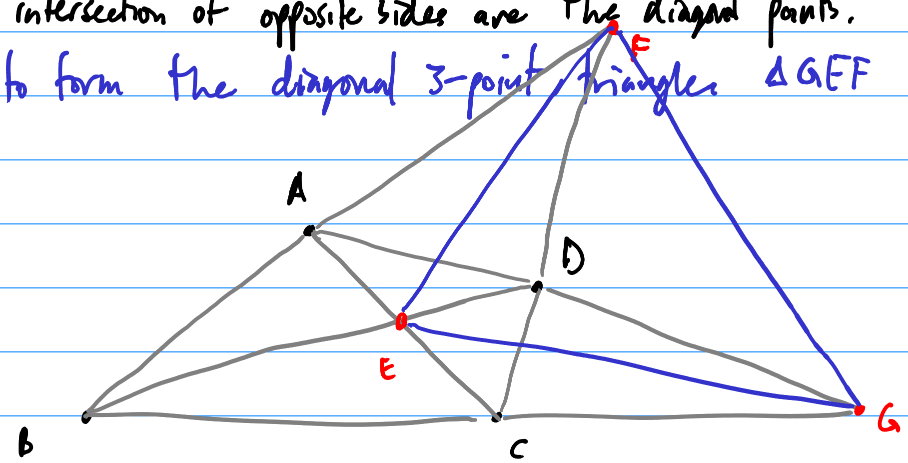
Def'n. 2.6.3 Complete Quadrangle
Choose A, B, C, D s.t. no 3 are collinear



The sides are the lines connecting the vertices.

The points of intersection of opposite sides are the diagonal points.
They connect to form the diagonal 3-point triangle ΔGEF

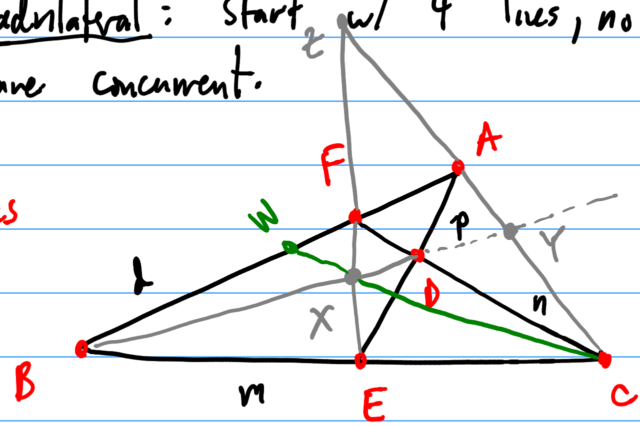
Another perspective:



Defn. Complete Quadrilateral: Start w/ 4 lines, no 3 of which are concurrent.

intersection pts = vertices

diagonal lines



Thm 2.617. On each diagonal line of a complete quadrilateral, there is a harmonic range of points: two vertices and two diagonal pts.

e.g., $(BD, XY) = -1$

To prove, consider the pencil of lines for which one of the other vertices is the vertex of the pencil

Apply Menelaus's Thm, then combine to get the cross ratio.

RE. Fill in the details. ∎