Ch 3 – Transformations of the Plane

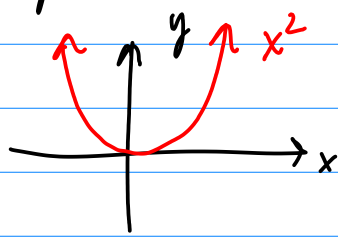
Defn. A transformation is a 1-1 function of a set X onto a set Y.

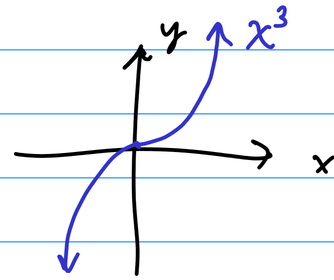
* **one-to-one:** if $f(x_1) = f(x_2)$, then $x_1 = x_2$.
"for each output there is exactly one input."

Ex. $y = x^2$ $\quad y = 4 \Rightarrow x = 2, -2$
NOT 1-1.

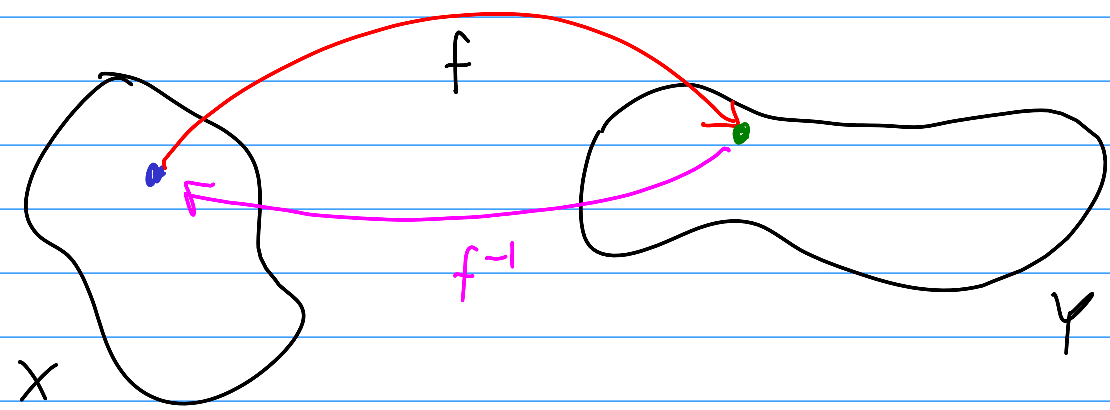
$y = x^3$ $\quad y = 8 \Rightarrow x = 2$

Horizontal Line Test:





* **Onto:** for every $y \in Y$, there exists an $x \in X$ such that $f(x) = y$.

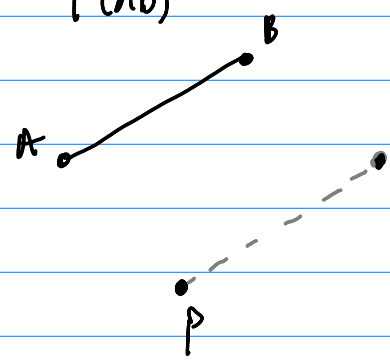


Defn: if $f: X \to Y$, but $X = Y$, then f maps X onto itself.
In this case, if $f(x) = x$, then x is called a fixed point of f.

Geometric Transformations: $f: \mathbb{R}^2 \to \mathbb{R}^2$

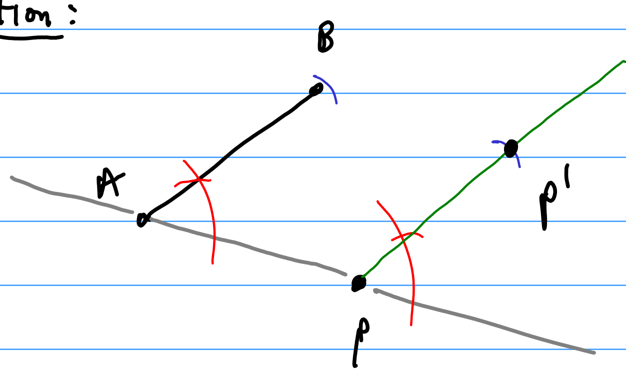
Ex. **Translations**: given a line segment $\overline{AB}$.

$T(AB)$

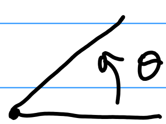


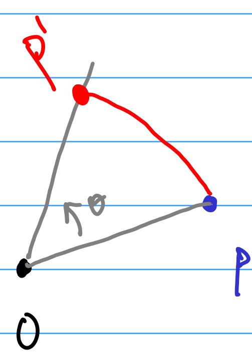
$P' = T(AB)(P) = T_{AB}(P)$

Construction:



Ex. **Rotation**: about a point O by an angle θ

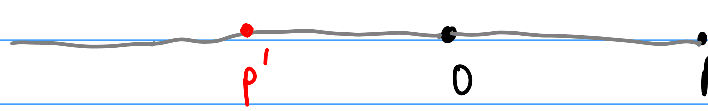




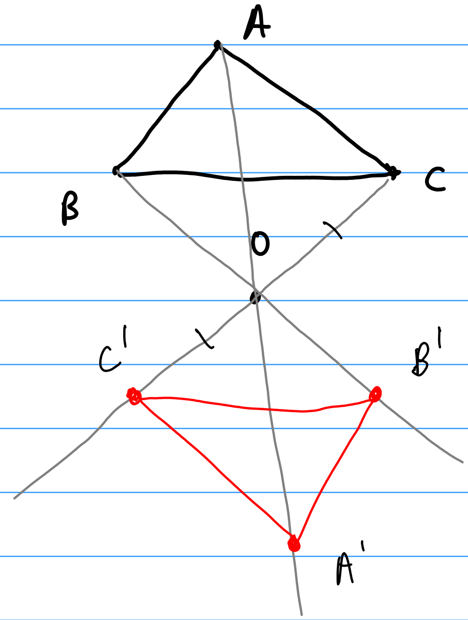
$R(O,\theta)(P) = R_{O,\theta}(P)$

Ex. $R(O,\pi)$ is a special rotation called a **reflection through the point** O.

$R_{O,\pi}(P) = R_O(P)$

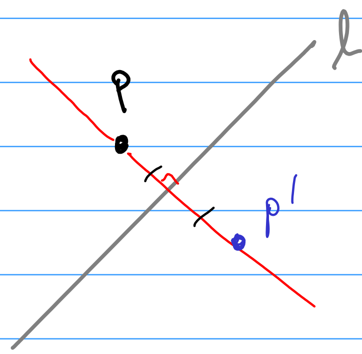


Ex. ΔABC. Find $R_O(\Delta ABC) = \Delta A'B'C'$

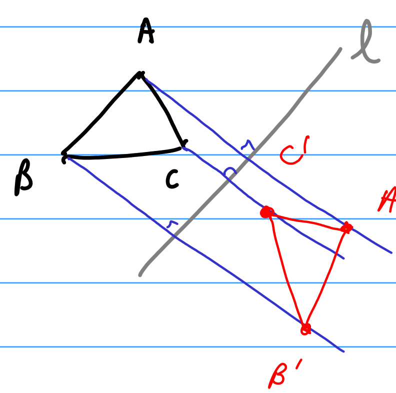


Orientation is preserved.

Ex. Reflection through a line: ℓ $\quad P' = R_\ell(P)$



Ex. $R_\ell(\Delta ABC) = \Delta A'B'C'$

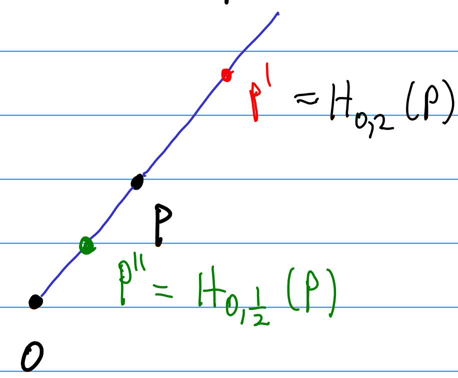


Reflections through ℓ reverse orientation.

- Translations, Rotations, and reflections are all isometries: angles, lengths, areas are all preserved.

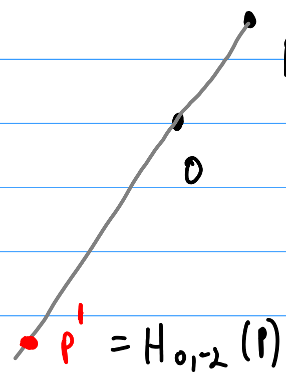
Ex. Homothety: $H(O,k)$ $\quad$ O is a point, k is a constant, $k \neq 0$.

$H_{(O,k)}$



$P' = H_{O,k}(P) \Rightarrow \overline{OP'} = k\,\overline{OP}$.

Ex. $H_{O,-2}(P)$



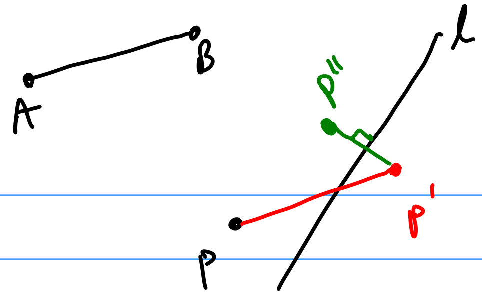
Ex. $H(O,-1) = R(O,\pi) = R(O)$.

→ if $|k| > 1$, then $H_{O,k}$ is a dilation

if $0 < |k| < 1$, then $H_{O,k}$ is a contraction.

Ex. $H_{O,1}$ = identity transformation (no changes!).

Ex. Glide Reflection: $G_{\ell,AB} = R(\ell) \circ T(AB) = R_\ell\left(T_{AB}(P)\right)$

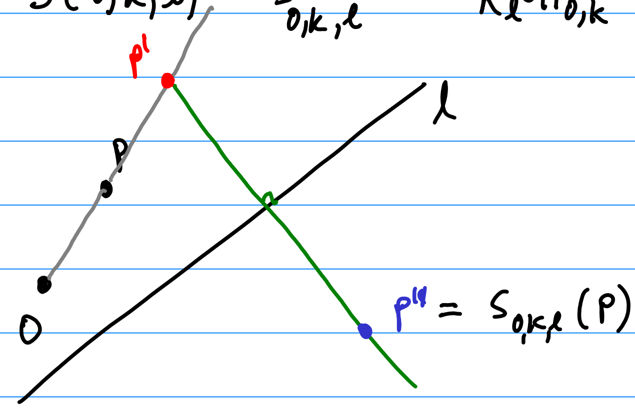


$P'' = G_{\ell, AB}(P)$

P' is an intermediate step.

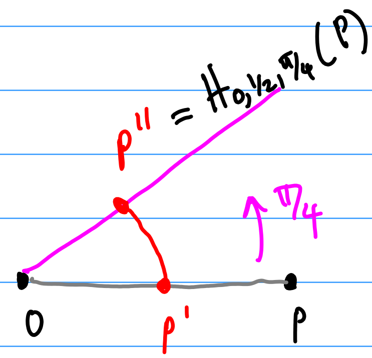
Ex. Dilation-Reflection: $S(0,k,\ell) = S_{0,k,\ell} = R_\ell \circ H_{0,k}$

$k=2$



Ex. Dilation-Rotation: $H(0,k,\theta) = H_{0,k,\theta} = R_{0,\theta} \circ H_{0,k}$

$H_{0,\frac{1}{2},\pi/4}$



Introduce some coordinates: $R_{0,\theta}$

$\mathbb{R}^2 = \mathbb{C}$

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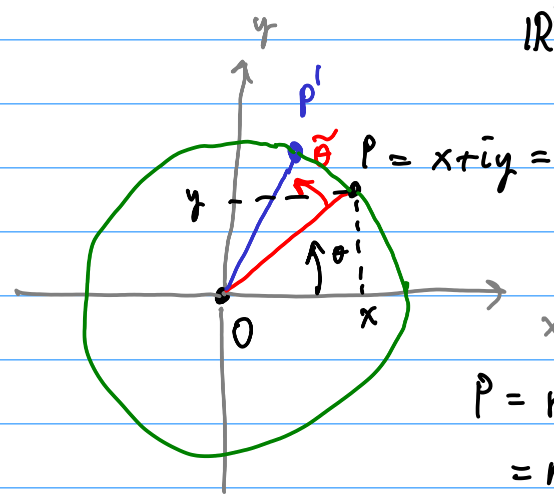
Describe P'



$P = x+iy = (x,y) = (r,\theta) = (r\cos\theta, r\sin\theta)$

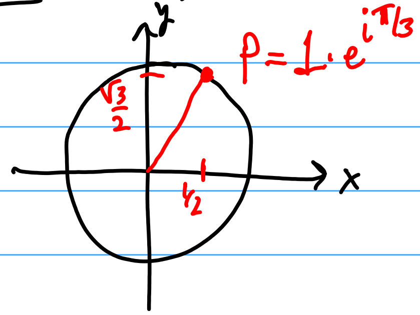
$r = \overline{OP}$

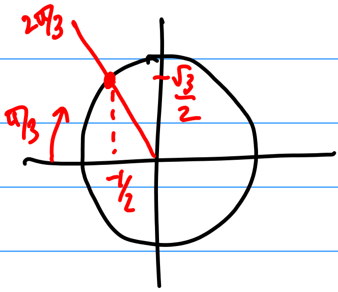
$\cos\theta + i\sin\theta = e^{i\theta}$

$P = r(\cos\theta + i\sin\theta)$
$= re^{i\theta}$

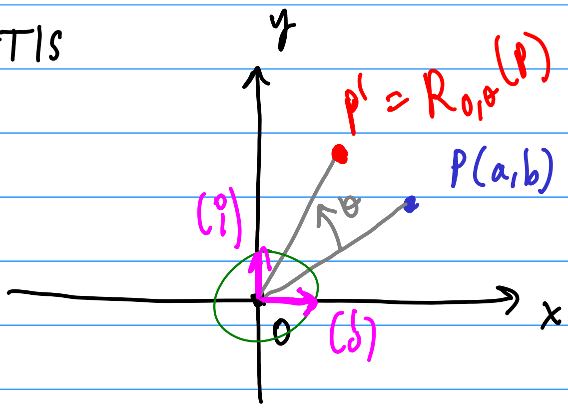
$$R_{0,\tilde{\theta}}(p) = re^{i\theta}\cdot e^{i\tilde{\theta}} = re^{i(\theta+\tilde{\theta})}$$

Ex. $p = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ $\qquad R_{0,\pi/3}(p) = ? = e^{i\pi/3}\cdot e^{i\pi/3} = e^{i(2\pi/3)} = -\frac{1}{2}+\frac{\sqrt{3}}{2}i$





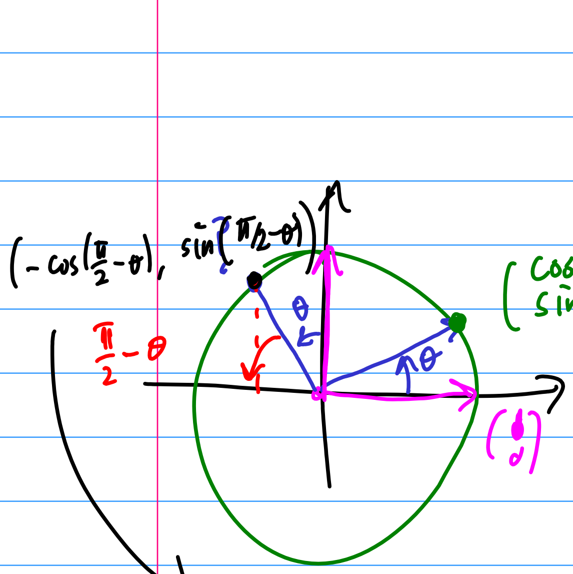
Ex. FTIS

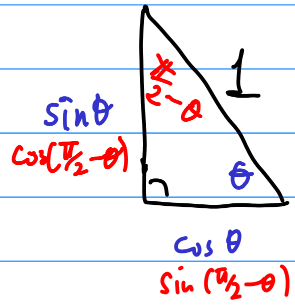


Find a matrix $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that

$T\begin{pmatrix}a\\b\end{pmatrix} = R_{0,\theta}(p)$. in coord's.

$$T = \left(\left[R_{0,\theta}\begin{pmatrix}1\\0\end{pmatrix}\right], \left[R_{0,\theta}\begin{pmatrix}0\\1\end{pmatrix}\right]\right)$$





Now apply cofunction IDs to get $\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}$ and $\boxed{T_\theta = \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}}$