M621 24 Oct '18

§3.3 – Isometries

Def'n. An isometry is a transformation of the plane to itself such that distances are preserved.

$$f: \mathbb{R}^2 \to \mathbb{R}^2 : \quad \overline{f(P)f(Q)} = \overline{PQ} = PQ$$

⊛ isometries do not need to preserve orientation.

$$d(f(P), f(Q)) = d(P, Q)$$

Isometries also preserve angles. "triangles to congruent triangles"

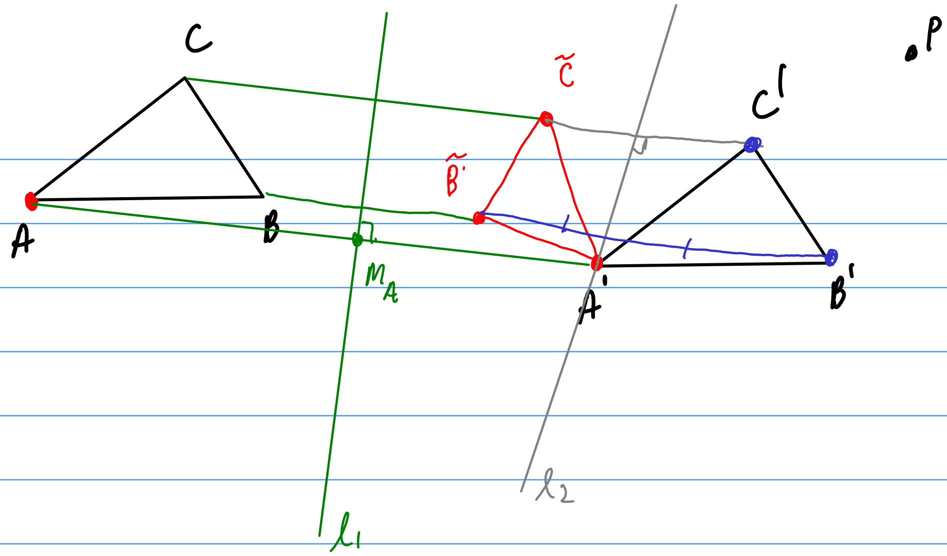
A transformation that preserves only angles, but not lengths is called conformal.

What kinds of transformations are isometries?

→ Translation : yes ✓
→ Rotation : yes ✓
→ Reflection : yes (does not preserve orientation) ✓
Homothety (Dilation): no.
→ Glide reflection: yes ✓

Thm. There is a unique isometry of the plane that maps a given triangle ΔABC to a given congruent triangle $\Delta A'B'C'$.

Proof. We will construct the isometry!



The isometry for this example is $R(\ell_2) \circ R(\ell_1)$

Thm. Any isometry of the plane can be represented by a composition of at most 3 reflections through lines.

Proof. Essentially the same construction.

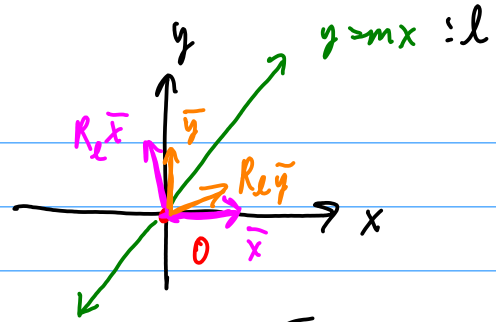
Thm. If f is an orientation-preserving isometry, then it can be decomposed as two reflections in lines.

Thm. If f has a fixed point $(f(p)=p)$ then f can be described by at most 2 reflections.

Thm. If f has 2 fixed points (distinct), then f is either the identity map or it's a reflection.

Thm. If f has 3 non-collinear fixed points, then f must be the identity function.

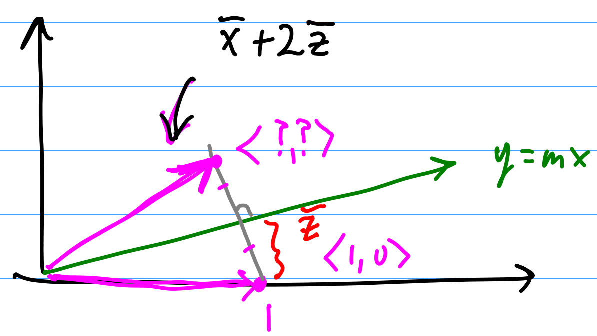
GP. 17.



Find the matrix (2x2) that represents the reflection over ℓ.

$$R_\ell = \left[\left[R_\ell \bar{x} \right], \left[R_\ell \bar{y} \right] \right]$$

Clean:



GP. 18. } $O(\mathbb{R}^2)$ = "Orthogonal Matrices in $\mathbb{R}^{2\times 2}$"

$$Q^{-1} = Q^T$$

$$Q = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow Q^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$\rightarrow Q^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

Show: Q is an isometry.

$$\bar{x} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad Q(\bar{x}) = Q\bar{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\|\bar{x}\| = \sqrt{\bar{x} \cdot \bar{x}} = \sqrt{(Q\bar{x}) \cdot (Q\bar{x})} = \|Q\bar{x}\|$$

use: $\bar{x} \cdot \bar{y} = \bar{x}^T \bar{y}$ hint: $(Q\bar{x})^T = \bar{x}^T Q^T$