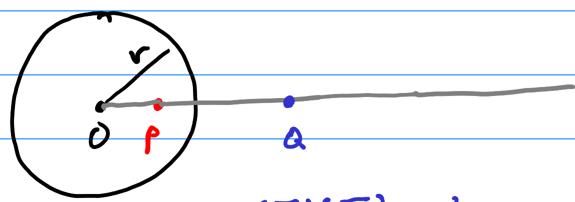
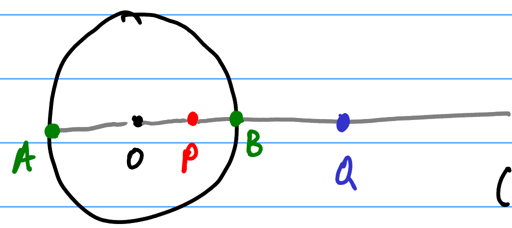
M621 5 Nov '18

## §3.4-5 Inversion




$(\overline{OP})(\overline{OQ}) = r^2$

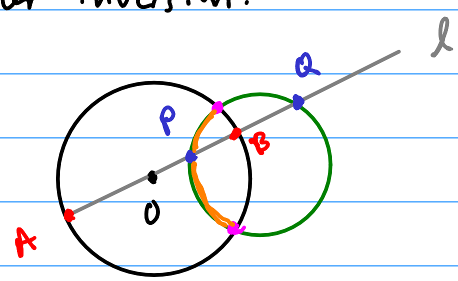
To construct, we use this fact:




$(AB, PQ) = -1$

Thm. A circle orthogonal to the circle of inversion is its own image under inversion.

"Proof".

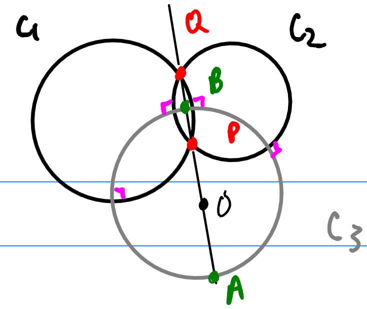



Thm from Ch2:
If $C_1 \perp C_2$, then any diameter of one circle cuts the other harmonically.

Thm. If C and D are inverse points wrt a circle $O(r)$, then any circle through CD is orthogonal to $O(r)$.

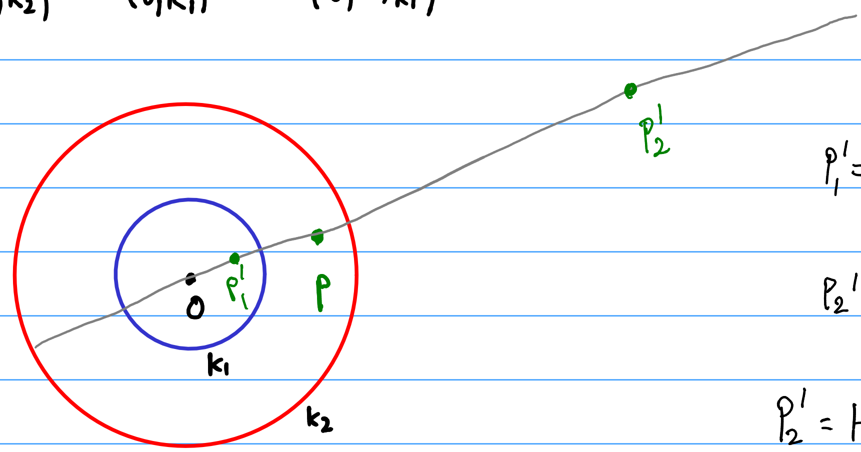
Thm. If two intersecting circles $C_1$ and $C_2$ are both orthogonal to a circle $C_3$, then the intersection pts of $C_1$ and $C_2$ are inverses wrt $C_3$.

"Proof".




$(AB, PQ) = -1$
since $AB$ is a diameter of $C_3$ cutting both $C_1$ and $C_2$ in the chord $AB$. Same Thm from Ch. 2.

**Thm.** $I_{(O,k_2)} \circ I_{(O,k_1)} = H_{(O,\, k_2^2/k_1^2)}$




$P_1' = I_{(O,k_1)}(P)$

$P_2' = I_{(O,k_2)}(P_1')$

$P_2' = H_{(O,\, k_2^2/k_1^2)}(P)$

Homothety defining property: $\underbrace{\overline{OP_2'} = \frac{k_2^2}{k_1^2}\overline{OP}}$ $\leftarrow$

**Proof.**
$\overline{OP} \cdot \overline{OP_1'} = k_1^2$
$\overline{OP_1'} \cdot \overline{OP_2'} = k_2^2 \longrightarrow \overline{OP_1'} = \frac{k_2^2}{\overline{OP_2'}}$, so $\overline{OP} \cdot \left(\frac{k_2^2}{\overline{OP_2'}}\right) = k_1^2$

so $\overline{OP_2'} = \frac{k_2^2}{k_1^2}\overline{OP}$ ∎ :)

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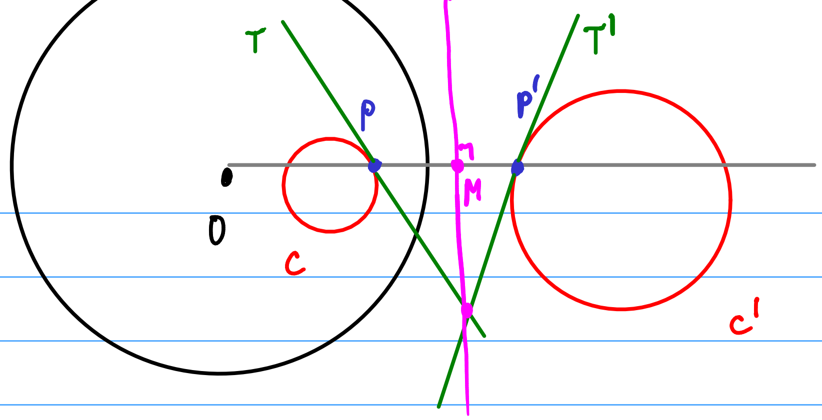
**Lemma.** Let $C'$ be the inverse of "circle" $C$, and let $P$ and $P'$ be a pair of corresponding points on $C$ and $C'$ respectively, such that $P$ and $P'$ are inverses of each other under the same inversion.
Then the tangents to $C$ and $C'$ through $P$ and $P'$ are reflections of one another through the line perpendicular to $OP$, passing through the midpoint of $PP'$.

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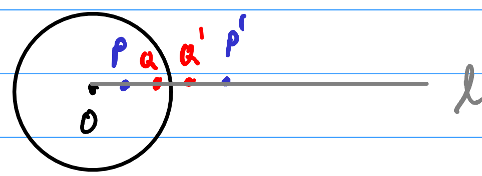
<u>Thm 3.4.13</u> If $(P,P')$ and $(Q,Q')$ are pairs of inverse pts wrt $I_{(O,r)}$, then

$$P'Q' = \frac{(PQ)r^2}{(OP)(OQ)}. \leftarrow$$

<u>Proof</u>. <u>Case I</u>. $O,P,Q$ are collinear so $P',Q'$ are also collinear w/ $O,P,Q$.




$$r^2 = \overline{OP}\cdot\overline{OP'} = \overline{OQ}\cdot\overline{OQ'}$$

$$(\overline{OQ}+\overline{QP})\cdot\overline{OP'} = \overline{OQ}\,(\overline{OP'}+\overline{P'Q'})$$

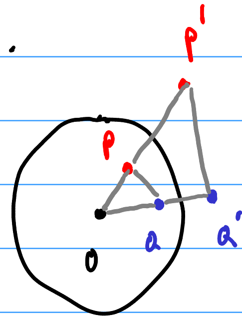
$$\cancel{\overline{OQ}\cdot\overline{OP'}} + \overline{QP}\cdot\overline{OP'} = \cancel{\overline{OQ}\cdot\overline{OP'}} + \overline{OQ}\cdot\overline{P'Q'}$$

$$\overline{P'Q'} = \frac{\overline{QP}\cdot\overline{OP'}}{\overline{OQ}}\cdot\frac{\overline{OP}}{\overline{OP}} = \frac{\overline{QP}\,(\overbrace{\overline{OP}\cdot\overline{OP'}}^{r^2})}{\overline{OQ}\cdot\overline{OP}}$$

Then, dropping the directions,

$$P'Q' = \frac{PQ\cdot r^2}{(OQ)\cdot(OP)} \quad \checkmark$$

<u>Case 2</u>. $O,P,Q$ not collinear.



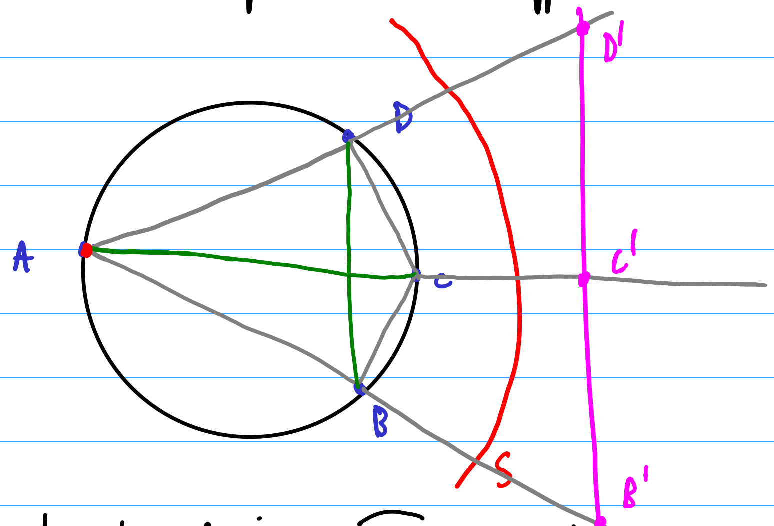

$r^2 = \overline{OP}\cdot\overline{OP'} = \overline{OQ}\cdot\overline{OQ'}$, therefore

$\Delta OPQ \sim \Delta OQ'P'$ (TBS).

Then $\dfrac{P'Q'}{PQ} = \dfrac{OQ'}{OP}$ and finish as above.

(Ptolemy)

<u>Thm.</u> In a cyclic convex quadrilateral the product of the diagonals is equal to the sums of the products of the opposite sides.




<u>"Proof"</u>. Choose $r >$ diameter of circle $\overset{\frown}{ABCD}$. Consider inversion $I_{(A,r)}$

$S =$ circle $A(r)$.

Invert the vertices $B, C, D$ through $I_{(A,r)}$. Then the images $B', C', D'$ are collinear.

Then,

$$B'D' = B'C' + C'D'.$$

Now Theorem 3.4.13 applies to each term, and

$$\frac{BD \cdot r^2}{AB \cdot AD} = \frac{BC \cdot r^2}{AB \cdot AC} + \frac{CD \cdot r^2}{AC \cdot AD}$$

Then multiply everything by $\dfrac{AB \cdot AC \cdot AD}{r^2}$

We get $\quad AC \cdot BD = AD \cdot BC + AB \cdot CD \quad \blacksquare$ :)