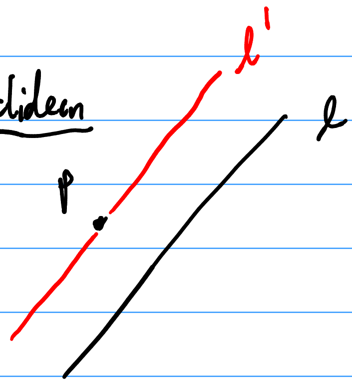
M621 — 7 Nov '18

Ch 4 Other (Non-Euclidean) Geometries

§ 4.3 Hyperbolic Geometry

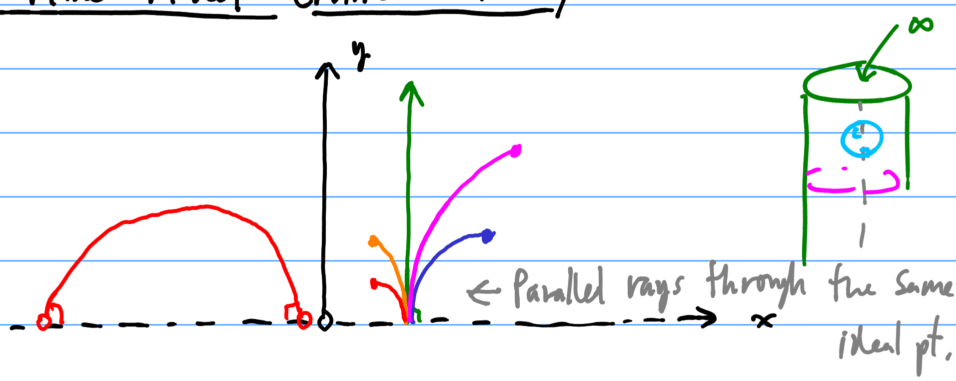
In hyperbolic geometry, given a line and a point not on the line, then there are infinitely many lines through the point parallel to the given line.

Euclidean



Hyperbolic Upper Half Plane Model (Poincaré Plane)

$\mathbb{H}$: $y > 0$

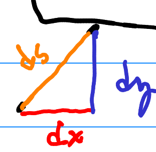


Regard the points at $y=0$ (x-axis) as ideal pts (at ∞).

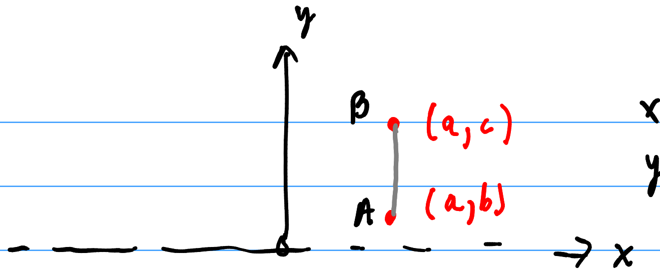
"lines" in $\mathbb{H}$ are semicircles that are orthogonal to the x-axis.

Distances in $\mathbb{H}$ are measured by $ds^2 = \dfrac{dx^2 + dy^2}{y^2}$ $\Rightarrow$ $\boxed{ds = \dfrac{\sqrt{dx^2+dy^2}}{y}}$

Euclidean Arc Length is $ds^2 = dx^2 + dy^2$



Ex.



$x=a$ $\quad\boxed{dx=0}$

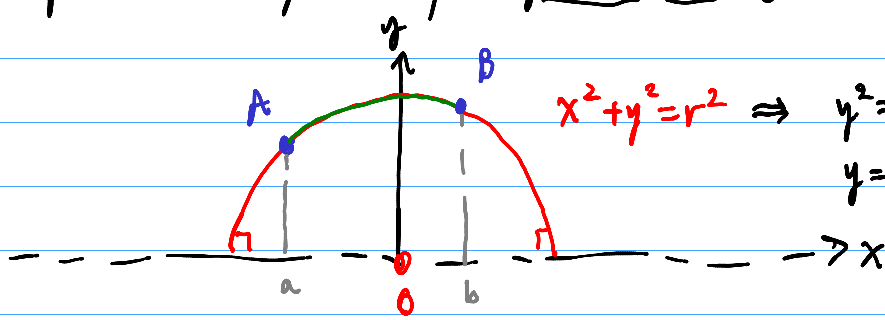
$y=y : b\to c$ $\quad dy=dy$

$$d(A,B)_{\mathbb{H}} = \ell = \int_{y=b}^{c} \frac{1}{y}\sqrt{\cancel{dx^2}+dy^2} = \int_b^c \frac{1}{y}\,dy = \ln(y)\Big|_b^c = \left|\ln(c)-\ln(b)\right|$$

$$\boxed{d(A,B)_{\mathbb{H}} = \left|\ln\left(\frac{c}{b}\right)\right|}$$ for vertical segments only.

$$\lim_{b\to 0^+} d(A,B)_{\mathbb{H}} = \lim_{b\to 0^+} \ln\left(\frac{c}{b}\right) = \ln\left(\lim_{b\to 0^+}\frac{c}{b}\right) = \ln(\infty) = \infty \checkmark$$

Ex. Compute the length along a geodesic arc ("line" in $\mathbb{H}$).



$x^2+y^2=r^2 \Rightarrow y^2=r^2-x^2$

$y=\sqrt{r^2-x^2}$

$y=\sqrt{r^2-x^2}$

$x=x$

$$\int_A^B \frac{1}{y}\underbrace{\sqrt{dx^2+dy^2}}$$

$$\frac{dy}{\cancel{dx}} = \frac{-\cancel{2}x\,dx}{\cancel{2}\sqrt{r^2-x^2}}$$

$$dx^2+dy^2 = dx^2 + \left(\frac{-x\,dx}{\sqrt{r^2-x^2}}\right)^2 = dx^2 + \frac{x^2\,dx^2}{r^2-x^2}$$

$$y^2 ds^2 = \left(\frac{r^2-x^2}{r^2-x^2} + \frac{x^2}{r^2-x^2}\right)dx^2 = \frac{r^2}{r^2-x^2}\,dx^2$$

$$\sqrt{dx^2+dy^2} = \frac{r}{\sqrt{r^2-x^2}}\,dx$$

So $$d(A,B)_{\mathbb{H}} = \int_a^b \frac{1}{\sqrt{r^2-x^2}}\cdot\frac{r}{\sqrt{r^2-x^2}}\,dx$$

$$= \int_a^b \frac{r \cdot \frac{1}{r^2}}{\frac{1}{r^2}(r^2 - x^2)}\, dx = \int_a^b \frac{1}{1-\left(\frac{x}{r}\right)^2} \cdot \underbrace{\frac{1}{r}\, dx}$$

$u = \frac{x}{r}$ $\quad u(b) = b/r$

$du = \frac{1}{r}dx$ $\quad u(a) = a/r$

$$= \int_{a/r}^{b/r} \frac{1}{1-u^2}\, du \overset{PFD}{=} \dots = \operatorname{arctanh} u \Big|_{a/r}^{b/r} = \tanh^{-1}(u)\Big|_{a/r}^{b/r}$$

$\sinh(x) = \frac{e^x - e^{-x}}{2}$

$\cosh(x) = \frac{e^x + e^{-x}}{2}$

$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$$\boxed{d(A,B)_{\mathbb{H}} = \tanh^{-1}(b/r) - \tanh^{-1}(a/r)}$$

$$\tanh^{-1}(u) = \underbrace{\frac{1}{2}\ln\left(\frac{1+u}{1-u}\right)}$$

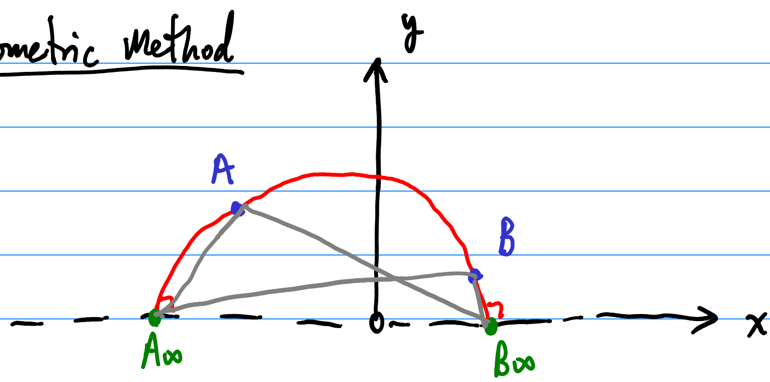
We get,

$$d(A,B)_{\mathbb{H}} = \frac{1}{2}\ln\left(\frac{1+b/r}{1-b/r}\right) - \frac{1}{2}\ln\left(\frac{1+a/r}{1-a/r}\right)$$

$$= \frac{1}{2}\ln\left(\frac{(1+b/r)(1-a/r)}{(1-b/r)(1+a/r)}\right)$$

$$\boxed{d(A,B)_{\mathbb{H}} = \ln\left(\frac{\sqrt{1+b/r}\,\sqrt{1-a/r}}{\sqrt{1-b/r}\,\sqrt{1+a/r}}\right)} \leftarrow \quad a, b: x\text{-components}$$

Euclidean Geometric Method



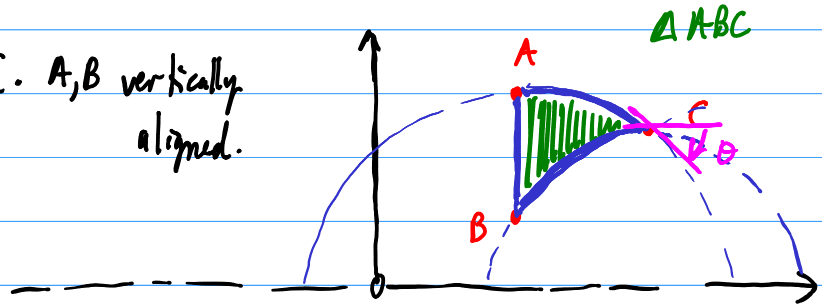
$$\boxed{d(A,B)_{\mathbb{H}} = \ln\left(\frac{\overline{AB_\infty} \cdot \overline{BA_\infty}}{\overline{AA_\infty} \cdot \overline{BB_\infty}}\right)}$$

where $\overline{PQ}$ = Euclidean length.

$$\overline{PQ} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Q. What do triangles look like? in $\mathbb{H}$?

Case I. A, B vertically aligned.



Sum of angles of triangles is less than $180° = \pi$.

Case II. No pairs of vertices lie on a common geodesic arc.