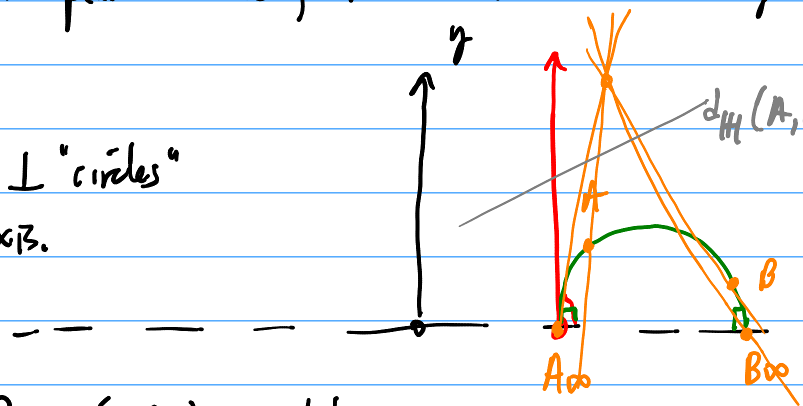
§4.3 - <u>Hyperbolic Geometry</u> : given a line ℓ and a point P not on ℓ, there exist at least two lines through P parallel to ℓ.

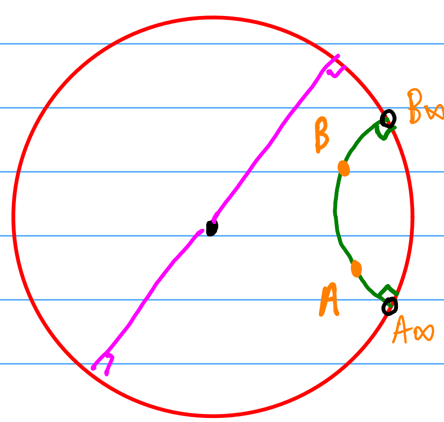
$\mathbb{H}$: upper half plane model, Poincaré Plane $y > 0$

lines are $\perp$ "circles" to x-axis.



$$d_{\mathbb{H}}(A,B) = \ln\left(\frac{\overline{AB_\infty}\cdot\overline{BA_\infty}}{\overline{AA_\infty}\cdot\overline{BB_\infty}}\right) = \ln\left(\frac{\overline{AB_\infty}}{\overline{BB_\infty}}\cdot\frac{\overline{BA_\infty}}{\overline{AA_\infty}}\right) = \ln(AB, B_\infty A_\infty)$$

<u>Ex</u>. $\mathbb{D}$ = Poincaré Disk model



Boundary represents points at ∞.

The center corresponds to the "other" pt at ∞. We call this the <u>pole</u>.

a diameter is a <u>line</u>: corresponding to vertical lines in $\mathbb{H}$.
Arcs of circles orthogonal to the boundary circle are lines in $\mathbb{D}$ as well.

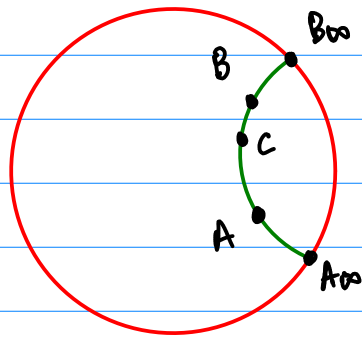
The <u>distance</u> between A, B in $\mathbb{D}$ is given by

$$d_{\mathbb{D}}(A,B) = d_{\mathbb{H}}(A,B) = \ln(AB, B_\infty A_\infty).$$

<u>Thm</u>. Let A, B, C be "collinear" in hyperbolic space. Then

$$d_{\mathbb{H}}(A,B) = d_{\mathbb{H}}(A,C) + d_{\mathbb{H}}(C,B).$$

In $\mathbb{D}$,



$$d_{\mathbb{H}}(A,B) = \ln(AB, B_\infty A_\infty) = \ln\left(\frac{AB_\infty}{B_\infty B} \cdot \frac{A_\infty B}{A\,A_\infty}\right)$$

$$= \ln\left(\frac{AB_\infty}{BB_\infty} \cdot \frac{BA_\infty}{AA_\infty} \cdot \frac{CB_\infty}{CA_\infty} \cdot \frac{CA_\infty}{CB_\infty}\right)$$

$$= \ln\left(\underbrace{\frac{AB_\infty}{CB_\infty} \cdot \frac{CA_\infty}{AA_\infty}} \cdot \underbrace{\frac{CB_\infty}{BB_\infty} \, \frac{BA_\infty}{CA_\infty}}\right)$$

$$= \ln(AC, \underbrace{B_\infty}_{C_\infty} A_\infty) + \ln(CB, B_\infty \underbrace{A_\infty}_{C_\infty})$$

$$= d_{\mathbb{H}}(A,C) + d_{\mathbb{H}}(C,B). \quad \blacksquare$$

Constructions in $\mathbb{D}$.