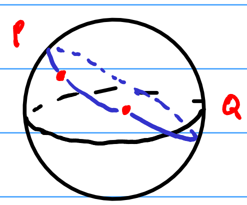
M621 14 Nov '18

§4.4 – Elliptic Geometry

→ Given a line ℓ and a point P not on ℓ, there exist no lines through P parallel to ℓ.

Models:

1. Riemann Sphere: unit sphere [embedded in $\mathbb{R}^3$.]

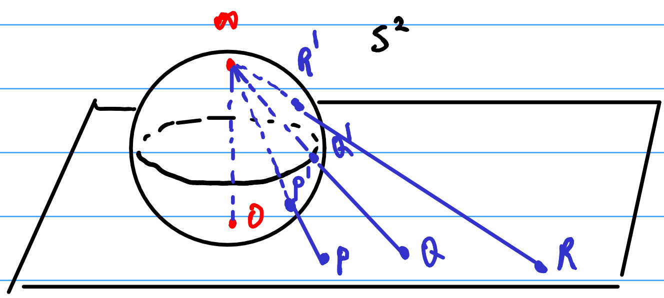


S^2 "lines" are equatorial circles

2. Extended Complex Plane, $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$

The antipodal pt to 0 is ∞

$\infty \longmapsto \{\infty\}$

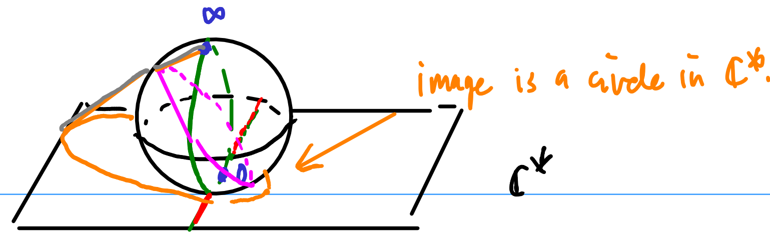


This process is called stereographic projection.

Q. What do "lines" look like in $\mathbb{C}^*$?

1. equators through 0 in S^2 map to lines through 0 in $\mathbb{C}^*$.

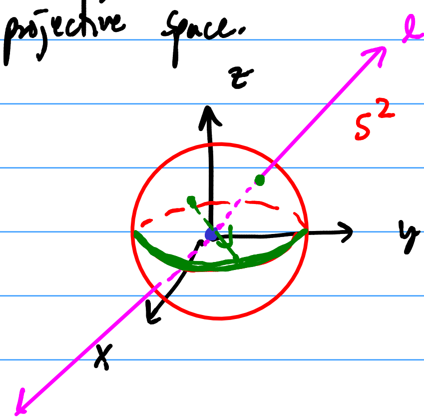
2. All other equators map to circles in $\mathbb{C}^*$.



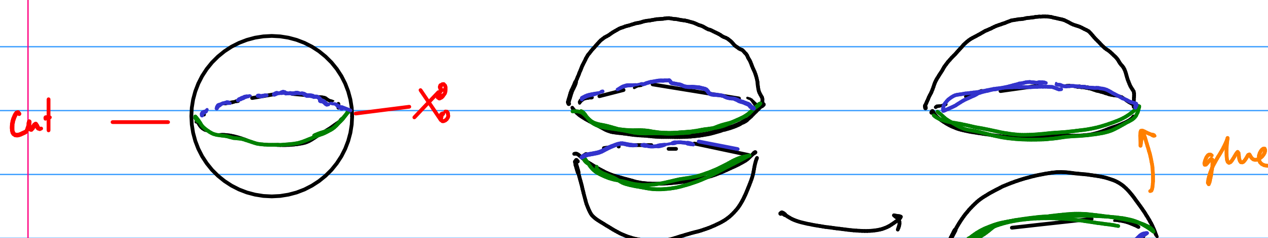
3. <u>Projective Plane</u> :

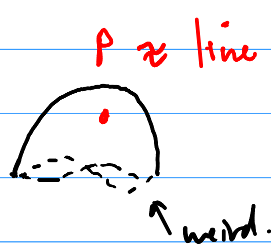
Consider the space $\mathbb{R}^3$.
Suppose we <u>identify</u> any two points in $\mathbb{R}^3$ for which $\vec{u} = \lambda\vec{u}$ for $\lambda \neq 0$.
Then $\vec{u}$ and $\lambda\vec{u}$ lie on a line through the origin in $\mathbb{R}^3$.

Identifying these points allows us to regard lines through 0 in $\mathbb{R}^3$ as single points in projective space.



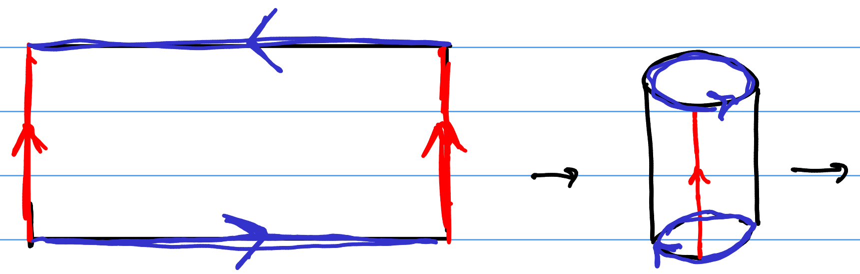
Antipodal pts of S^2 are identified.

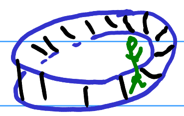




$\mathbb{R}P^2$ real projective plane.

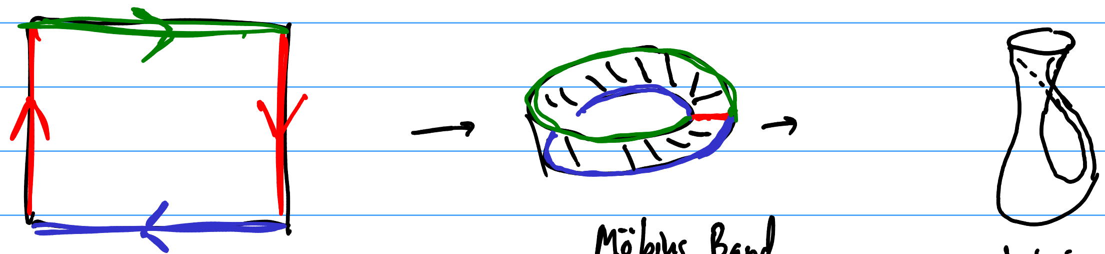
Another way to construct $\mathbb{R}P^2$.





Möbius strip

Ex.

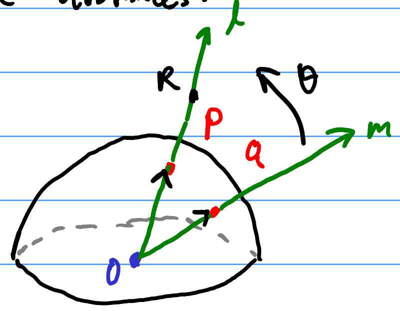


Möbius Band

Klein Bottle $\mathbb{K}l^2$

How do we measure distances?

In $\mathbb{R}P^2$,



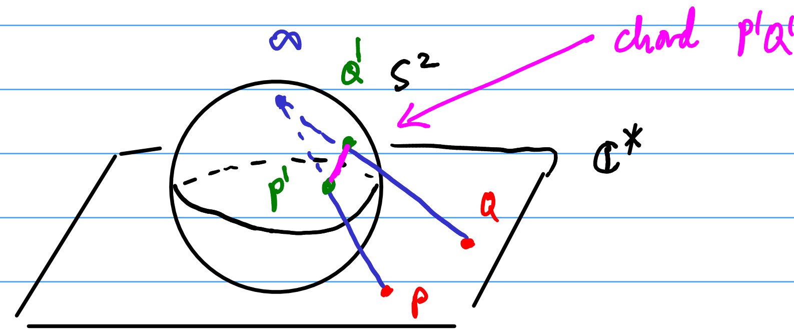
$P, Q \in \mathbb{R}P^2$

$$d_{\mathbb{E}}(P,Q) = |m \measuredangle POQ|$$

If we regard P, Q as pts in $\mathbb{R}^3$ lying on S^2, then they may also be regarded as vectors, and

$$\boxed{d_{\mathbb{E}}(P,Q) = \cos^{-1}\left(\frac{|P \cdot Q|}{\|P\| \, \|Q\|}\right)}$$

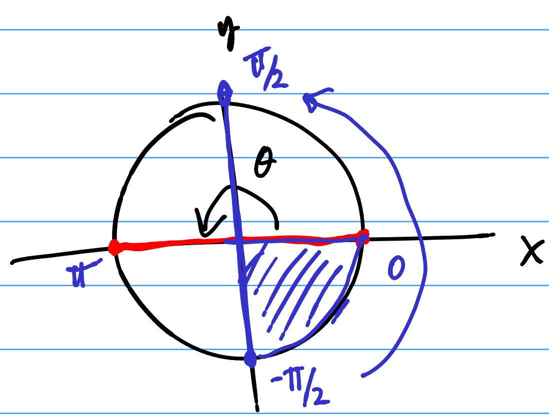
In $\mathbb{C}^*$ and S^2, the chordal distance P and Q in $\mathbb{C}^*$ is defined to be the length of the chord $\overline{P'Q'}$ in $\mathbb{R}^3$ that connects P' to Q' <u>inside</u> of S^2.



chordal distance:
$$\delta(P,Q) = \frac{2\,\|P-Q\|}{\sqrt{(1+\|P\|^2)}\sqrt{(1+\|Q\|^2)}}$$

and
$$\delta(P,\infty) = \frac{2\,\|P\|}{\sqrt{1+\|P\|^2}}$$

The elliptic distance is then given by
$$\boxed{d_E(P,Q) = 2\sin^{-1}\left(\frac{\delta(P,Q)}{2}\right)}$$



Trig. can be done to show that these formulas are "the same".