Name: Key
M511: **Linear Algebra** (Spring 2018)
Instructor: Justin Ryan
Unit I Exam: Chapters 3 and 4

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Instructions. *You must complete problems 1, 2, and 3. If you wish, you may omit one of problems 4, 5, or 6. If all 6 problems are to be graded, then each problem is worth 15 points and the minimum score is 10. If you choose to omit a problem, then each problem is worth 20 points and the minimum score is 0. Please initial one option below.*

______ *I would like all 6 problems to be graded.*

______ *I have omitted one of problems 4, 5, or 6.*

Part I. True/False *Neatly write* **T** *on the line if the statement is always true, and* **F** *otherwise. In the space provided below the statement, give sufficient explanation of your answer.*

F **1.a.** Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ forms a basis of a vector space V and $\mathbf{x} \in V$. Then $\{\mathbf{x}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ also forms a basis of V.

$\bar{x} = c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3$, c_i not all 0, so $\{\bar{x}, \bar{v}_1, \bar{v}_2, \bar{v}_3\}$ are l.d.

or $\bar{x} = \bar{0}$ so $\{\bar{0}, \bar{v}_1, \bar{v}_2, \bar{v}_3\}$ are l.d. (see #6a.)

T **1.b.** The vectors $\cos(t)$ and $\sin(t)$ are linearly independent in $C^1(\mathbb{R})$.

$$W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1.$$

T **1.c.** The subset $S = \{(x, y)^t \in \mathbb{R}^2 \mid y = 2x\}$ forms a subspace of $\mathbb{R}^2$.

$\bar{x} = \begin{pmatrix} x \\ 2x \end{pmatrix}$ $\quad \bar{x} + \bar{y} = \begin{pmatrix} x+y \\ 2(x+y) \end{pmatrix}$. ✓

$\alpha\bar{x} = \begin{pmatrix} \alpha x \\ 2\alpha x \end{pmatrix}$ ✓

F **1.d.** The function defined by $L(a + bi) = a^2 + b^2$ is a linear transformation $L : \mathbb{C} \to \mathbb{R}$.

$$L(\alpha(a+bi)) = \alpha^2 a^2 + \alpha^2 b^2 = \alpha^2 L(a+bi) \neq \alpha L(a+bi).$$

F **1.e.** If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ form a spanning set for a vector space V, then the dimension of V is 3.

$\bar{u}_i$ could be l.d.

Part II. *Complete both problems, showing enough work.*

Consider the functions $u_1(t) = e^t$ and $u_2(t) = te^t$.

2. Show that u_1 and u_2 are linearly independent in $C^\infty(\mathbb{R})$.

$$w(e^t, te^t) = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t} + te^{2t} - te^{2t} = e^{2t} \neq 0.$$

Since the Wronskian is not 0, the functions are linearly independent.

3. Consider the vector space $S = \text{span}\{u_1, u_2\}$. Find the matrix representations of the linear transformations $D : S \to S$ and $J : S \to S$ defined by

$$\begin{cases} D(f) = f'(t), \text{ and} \\ J(f) = \int f(t)\,dt. \end{cases}$$

$$D(e^t) = e^t = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \rightarrow \qquad D = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
$$D(te^t) = e^t + te^t = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$J(e^t) = \int e^t dt = e^t = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \rightarrow \qquad J = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$
$$J(te^t) = \int te^t\,dt = te^t - e^t = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\underline{\text{or}} \qquad J = D^{-1} = \frac{1}{\det(D)} C_D^t = \frac{1}{1} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Part III. *You may omit one of the following questions if you'd like, although you are not required to do so. (See the instructions on page 1.) If you do choose to make an omission, clearly indicate which problem you would like to be omitted.*

4. Suppose $L : \mathbb{R}^2 \to \mathbb{R}^3$ is given by

$$L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \\ x_1 + x_2 \end{pmatrix}.$$

a.) Write the matrix A representing L with respect to the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$.

$L(\bar{e}_1) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$L(\bar{e}_2) = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ $\Rightarrow$ $\boxed{A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}}$

b.) Find the row and null spaces of A.

$\boxed{\text{Row}(A) = \text{span}\left\{ \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}.}$

Null: $\left(\begin{array}{cc|c} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = 0 \end{array}$, so $\boxed{\text{Null}(A) = \{\bar{0}\}.}$

c.) Verify that your answer to part *b.*) obeys the Rank-Nullity Theorem.

$\dim(\text{Row}(A)) = 2 = \text{rk}(A)$

$\dim(\text{Null}(A)) = 0 = \text{null}(A)$

$\text{rk}(A) + \text{null}(A) = 2 + 0 = 2 = \dim(\mathbb{R}^2)$ ✓

5. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

a.) Write down all minors of A. Clearly label each minor.

$$M_{11} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \qquad M_{12} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \qquad M_{13} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$M_{21} = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix} \qquad M_{22} = \begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix} \qquad M_{23} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$M_{31} = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} \qquad M_{32} = \begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix} \qquad M_{33} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$$

b.) Use the cofactor method to compute A^{-1}. You must use the cofactor method to receive credit.

$$C_A = \begin{pmatrix} +(-1) & -1 & +(-1) \\ -(-5) & +(-1) & -1 \\ +2 & -4 & +2 \end{pmatrix} = \begin{pmatrix} -1 & -1 & -1 \\ 5 & -1 & -1 \\ 2 & -4 & 2 \end{pmatrix}$$

$$\det(A) = -1(1) - 1(2) - 1(3) = -6$$

$$A^{-1} = \frac{1}{\det(A)} C_A^t = \frac{-1}{6} \begin{pmatrix} -1 & 5 & 2 \\ -1 & -1 & -4 \\ -1 & -1 & 2 \end{pmatrix}$$

or

$$\boxed{A^{-1} = \begin{pmatrix} 1/6 & -5/6 & -1/3 \\ 1/6 & 1/6 & 2/3 \\ 1/6 & 1/6 & -1/3 \end{pmatrix}}$$

6. Let $(V, \oplus, \otimes)$ be a vector space. Prove the following statements.

a.) Any collection of vectors in V that contains the $\mathbf{0}$-vector is linearly dependent.

Let $\mathcal{U} = \{\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_n, \bar{0}\}$.

Then $0\bar{u}_1 + 0\bar{u}_2 + \ldots + 0\bar{u}_n + 1 \cdot \bar{0} = \bar{0}$

since all coefficients are not 0, then $\mathcal{U}$ is linearly dependent. ∎

b.) Let $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$. The subset $S = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a subspace of V.

Any elements in S can be written as

$$\bar{x} = x_1\bar{v}_1 + x_2\bar{v}_2 + \cdots + x_n\bar{v}_n$$

and $$\bar{y} = y_1\bar{v}_1 + y_2\bar{v}_2 + \cdots + y_n\bar{v}_n$$

Then,

$\underline{S1}$. $(\alpha\bar{x}) = \alpha(x_1\bar{v}_1 + x_2\bar{v}_2 + \cdots + x_n\bar{v}_n) = \alpha x_1\bar{v}_1 + \alpha x_2\bar{v}_2 + \ldots + \alpha x_n\bar{v}_n$

$= (\alpha x_1)\bar{v}_1 + (\alpha x_2)\bar{v}_2 + \cdots + (\alpha x_n)\bar{v}_n$

is a linear combination of $\bar{v}_i$'s.

Hence $(\alpha\bar{x}) \in S$.

S2. $(\bar{x} + \bar{y}) = (x_1\bar{v}_1 + x_2\bar{v}_2 + \cdots + x_n\bar{v}_n) + (y_1\bar{v}_1 + y_2\bar{v}_2 + \cdots + y_n\bar{v}_n)$

$= (x_1+y_1)\bar{v}_1 + (x_2+y_2)\bar{v}_2 + \ldots + (x_n+y_n)\bar{v}_n$

is also a linear combination of $\bar{v}_i$'s.

Hence, $(\bar{x} + \bar{y}) \in S$.

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