M511 - Linear Algebra   Final Exam Review Guide   Brief Solutions   22 Jul '18

1.) $E=\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$, $\quad V=\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}-3\\1\\0\end{pmatrix},\begin{pmatrix}9\\-6\\1\end{pmatrix}\right\}$

$S: V\to E = V = \begin{pmatrix}1&-3&9\\0&1&-6\\0&0&1\end{pmatrix}$ $\qquad S^{-1}: E\to V = V^{-1} = \begin{pmatrix}1&3&9\\0&1&6\\0&0&1\end{pmatrix}$

$$\left(\begin{array}{ccc|ccc}1&-3&9&1&0&0\\0&1&-6&0&1&0\\0&0&1&0&0&1\end{array}\right)\to\left(\begin{array}{ccc|ccc}1&-3&0&1&0&-9\\0&1&0&0&1&6\\0&0&1&0&0&1\end{array}\right)\to\left(\begin{array}{ccc|ccc}1&0&0&1&3&9\\0&1&0&0&1&6\\0&0&1&0&0&1\end{array}\right)$$

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b.) $D(1)=0$

$D(x-3)=1$

$D((x-3)^2) = 2x-6 = 2(x-3)$

so, $D=\begin{pmatrix}0&1&0\\0&0&2\\0&0&0\end{pmatrix}$

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c.) $\langle 1,1\rangle = 1^2+\frac{1}{2}1^2+1^2 = 2+\frac{1}{2} = 5/2$

$\langle 1,x\rangle = 1(-1)+\frac{1}{2}(1)(0)+1(1) = -1+0+1=0$

$\langle 1,x^2\rangle = 1(1)+\frac{1}{2}(1)(0)+1(1) = 2$

$\langle x,x\rangle = -1^2+\frac{1}{2}(0)^2+1^2 = 2$

$\langle x,x^2\rangle = -1\cdot 1+\frac{1}{2}(0)^2+1\cdot 1 = 0$

$\langle x^2,x^2\rangle = 1+\frac{1}{2}(0)+1 = 2$

so, $G=\begin{pmatrix}5/2&0&2\\0&2&0\\2&0&2\end{pmatrix}$

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d.) $\bar r_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$

$\|\bar r_1\| = \sqrt{5/2}$

$\bar u_1 = \sqrt{2/5}\begin{pmatrix}1\\0\\0\end{pmatrix}$

$\bar p_2 = \langle \bar x_2,\bar u_1\rangle \bar u_1 = \bar 0$

$\bar r_2 = \bar x_2 - \bar p_2 = \bar x_2$

$\|\bar r_2\| = \sqrt 2$

$\bar u_2 = \frac{1}{\sqrt2}\begin{pmatrix}0\\1\\0\end{pmatrix}$

$\bar p_3 = \langle \bar x_3,\bar u_1\rangle\bar u_1 + \langle \bar x_3,\bar u_2\rangle \bar u_2 = 2\sqrt{\frac{2}{5}}^2\begin{pmatrix}1\\0\\0\end{pmatrix} + 0 = \frac{4}{5}\begin{pmatrix}1\\0\\0\end{pmatrix}$

$\bar r_3 = \bar x_3 - \bar p_3 = \begin{pmatrix}0\\0\\1\end{pmatrix} - \begin{pmatrix}4/5\\0\\0\end{pmatrix} = \begin{pmatrix}-4/5\\0\\1\end{pmatrix} = \bar x_3 - 4/5\,\bar x_1$

$\|\bar r_3\|^2 = g_{33} - \frac{8}{5}g_{13} + \frac{16}{25}g_{11} = 2-\frac{16}{5}+\frac{16}{25}\cdot\frac{5}{2}$

$= 2 - 8/5 = 2/5$

so $\bar u_3 = \sqrt{\frac{5}{2}}\begin{pmatrix}-4/5\\0\\1\end{pmatrix}$

The orthonormal basis is: $U=\left\{\sqrt{\frac{2}{5}},\ \frac{1}{\sqrt2}x,\ \sqrt{\frac{5}{2}}x^2 - 2\sqrt{\frac{2}{5}}\right\}$

2.

x	-2	-1	0	1
y	0	1	4	5

$mx+b=y$

$-2m+b=0$

$-1m+b=1$

$0m+b=4$

$1m+b=5$

$A=\begin{pmatrix}-2&1\\-1&1\\0&1\\1&1\end{pmatrix}$ $\qquad A^T=\begin{pmatrix}-2&-1&0&1\\1&1&1&1\end{pmatrix}$

$A^TA=\begin{pmatrix}6&-2\\-2&4\end{pmatrix}$ $\qquad {A^TA}^{-1}=\frac{1}{20}\begin{pmatrix}4&2\\2&6\end{pmatrix}$

$A^T\bar b = \begin{pmatrix}4\\10\end{pmatrix}$

$\hat x = \frac{1}{20}\begin{pmatrix}4&2\\2&6\end{pmatrix}\begin{pmatrix}4\\10\end{pmatrix} = \frac{1}{20}\begin{pmatrix}36\\68\end{pmatrix}$

so the line is $y=\frac{36}{20}x+\frac{68}{20} = 1.8x+3.4$

3. $A-\lambda I = \begin{pmatrix} 4-\lambda & -5 & 1 \\ 1 & -\lambda & -1 \\ 0 & 1 & -1-\lambda \end{pmatrix}$ $\quad |A-\lambda I| = 4-\lambda \begin{vmatrix} -\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} - 1 \begin{vmatrix} -5 & 1 \\ 1 & -1-\lambda \end{vmatrix}$

$= (4-\lambda)\left[-\lambda(-1-\lambda)+1\right] - \left(-5(-1-\lambda)-1\right)$

$= 4(\lambda-4)\left[\lambda^2+\lambda+1\right] + \left[5\lambda+4\right]$

$= \left(\lambda^3+\lambda^2+\lambda-4\lambda^2-4\lambda-4+5\lambda+4\right)$

$= \lambda^3-3\lambda^2+2\lambda = \lambda(\lambda-2)(\lambda-1)$ $\quad \boxed{\lambda = 0,1,2}$

$\lambda=0$: $\left(\begin{array}{ccc|c} 4 & -5 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $\quad x_1 = t$, $x_2 = t$, $x_3 = t$ $\quad \boxed{\bar{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \lambda_1 = 0}$

$\lambda=1$: $\left(\begin{array}{ccc|c} 3 & -5 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $\quad x_1 = 3t$, $x_2 = 2t$, $x_3 = t$ $\quad \boxed{x_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \lambda_2 = 1}$

$\lambda=2$: $\left(\begin{array}{ccc|c} 2 & -5 & 1 & 0 \\ 1 & -2 & -1 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & -1 & 3 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $\quad x_1 = 5t$, $x_2 = 3t$, $x_3 = t$ $\quad \boxed{x_3 = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix}, \lambda_3 = 2}$

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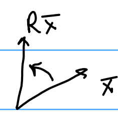
4. $|R-\lambda I| = (\cos\theta-\lambda)^2 + \sin^2\theta$ $\qquad$ but $0 \le \cos^2\theta \le 1$ and only $=1$ if $\theta = n\pi$

$= \lambda^2 - 2\cos\theta\,\lambda + 1$ $\qquad$ therefore λ is complex if $\theta \ne n\pi$.

so $\lambda = \cos\theta \pm \sqrt{\cos^2\theta - 1}$

This is a rotation matrix



It will only map $\bar{x}$ to a multiple of $\bar{x}$ if θ is $0, \pi, 2\pi$, etc.

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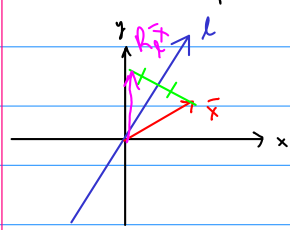
5. These are all proved in the Good Problems (#5?)

6.



$\bar{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\bar{y} = \begin{pmatrix} m \\ 1 \end{pmatrix}$ $\qquad \bar{p} = \dfrac{mx+y}{m^2+1}\begin{pmatrix} m \\ 1 \end{pmatrix}$

$\bar{z} = \bar{x} - \bar{p} = \begin{pmatrix} x - \frac{1}{m^2+1}(m^2x+my) \\ y - \frac{1}{m^2+1}(mx+y) \end{pmatrix}$

$R_\ell\bar{x} = \bar{x} - 2\bar{z} = \begin{pmatrix} x \\ y \end{pmatrix} - 2\begin{pmatrix} x - \frac{m}{m^2+1}(mx+y) \\ y - \frac{1}{m^2+1}(mx+y) \end{pmatrix}$

$= \begin{pmatrix} -x + \frac{2m}{m^2+1}(mx+y) \\ -y + \frac{2}{m^2+1}(mx+y) \end{pmatrix}$

$R_\ell\bar{e}_1 = \begin{pmatrix} -1 + \frac{2m^2}{m^2+1} \\ \frac{2m}{m^2+1} \end{pmatrix} = \begin{pmatrix} \frac{m^2-1}{m^2+1} \\ \frac{2m}{m^2+1} \end{pmatrix}$

$R_\ell\bar{e}_2 = \begin{pmatrix} \frac{2m}{m^2+1} \\ -1 + \frac{2}{m^2+1} \end{pmatrix} = \begin{pmatrix} \frac{2m}{m^2+1} \\ \frac{-(m^2-1)}{m^2+1} \end{pmatrix}$

so,

$$\boxed{R_\ell = \frac{1}{m^2+1}\begin{pmatrix} m^2-1 & 2m \\ 2m & -(m^2-1) \end{pmatrix}}$$

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7. $A = \begin{pmatrix} 0 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$ $\qquad \ker(L) = \text{Null}(A)$: $\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right)$ $\quad x_1 = t$, $x_2 = t$, $x_3 = 0$ $\qquad \ker(L) = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right\}$.

$\text{Im}(L) = \text{Col}(A) = \mathbb{R}^2$.

8. $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ $\quad A = \begin{pmatrix} 2 & -3 \\ 4 & 6 \end{pmatrix}$ $\quad p(\lambda) = \lambda^2 - 8\lambda + 24 = (\lambda - 4)^2 + 8$

$\lambda = 4 \pm 2\sqrt{2}\, i \qquad a = 4,\ b = 2\sqrt{2}$

$$\left(\begin{array}{cc|c} 2-(4+2\sqrt{2}\,i) & -3 & 0 \\ 4 & 6-(4+2\sqrt{2}\,i) & 0 \end{array}\right) \to \left(\begin{array}{cc|c} -2-2\sqrt{2}\,i & -3 & 0 \\ 4 & 2-2\sqrt{2}\,i & 0 \end{array}\right) \qquad x_1 = \frac{3}{-2-2\sqrt{2}\,i}\, t = \frac{(-6+6\sqrt{2}\,i)t}{12} = \left(\frac{-1}{2} + \frac{1}{\sqrt{2}} i\right)t, \quad x_2 = t$$

$$\frac{4}{-2-2\sqrt{2}\,i} = \frac{-8+8\sqrt{2}\,i}{4+8} = \frac{-8}{12} + \frac{8\sqrt{2}}{12} i$$

$$\vec{x} = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$-3\left(-\frac{8}{12} + \frac{8\sqrt{2}}{12} i\right) = \frac{8}{4} - \frac{8\sqrt{2}}{4} i = 2 - 2\sqrt{2}\, i \ ✓ \quad :)$$

The soln is: $\quad Y(t) = c_1 e^{4t} \cos(2\sqrt{2}t) \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} - c_1 e^{4t} \sin(2\sqrt{2}t) \begin{pmatrix} 1/\sqrt{2} \\ 0 \end{pmatrix} + c_2 e^{4t} \cos(2\sqrt{2}t) \begin{pmatrix} 1/\sqrt{2} \\ 0 \end{pmatrix} + c_2 e^{4t} \sin(2\sqrt{2}t) \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$

or $\quad \begin{cases} y_1(t) = \left(-\frac{1}{2}c_1 + \frac{1}{\sqrt{2}} c_2\right) e^{4t} \cos(2\sqrt{2}t) + \left(-\frac{1}{\sqrt{2}} c_1 - \frac{1}{2} c_2\right) e^{4t} \sin(2\sqrt{2}t) \\ y_2(t) = c_1 e^{4t} \cos(2\sqrt{2}t) + c_2 e^{4t} \sin(2\sqrt{2}t) \end{cases}$

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9. $Y = \begin{pmatrix} y \\ y' \end{pmatrix}$ $\quad Y' = \begin{pmatrix} y' \\ y'' \end{pmatrix}$ $\quad A = \begin{pmatrix} 0 & 1 \\ -4 & 5 \end{pmatrix}$ $\quad |A - \lambda I| = \lambda(\lambda - 5) + 4 = \lambda^2 - 5\lambda + 4 \quad \lambda = 1, 4$

$\lambda = 1$: $\left(\begin{array}{cc|c} -1 & 1 & 0 \\ -4 & 4 & 0 \end{array}\right)$ $\quad \vec{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\qquad \lambda = 4$: $\left(\begin{array}{cc|c} -4 & 1 & 0 \\ -4 & 1 & 0 \end{array}\right)$ $\quad \vec{x}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

$Y = c_1 e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 1 \\ 4 \end{pmatrix} \Rightarrow y(t) = c_1 e^{t} + c_2 e^{4t}$

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10. $\lambda = 1, -1$

$\lambda = 1$: $\left(\begin{array}{cc|c} 0 & 2 & 0 \\ 0 & -2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\quad x_1 = t,\ x_2 = 0 \quad \vec{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda = -1$: $\left(\begin{array}{cc|c} 2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\quad x_1 = -t,\ x_2 = t \quad \vec{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$X = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ $\quad X^{-1} = \frac{1}{-1}\begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ $\quad D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad e^{D} = \begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix}$

$$e^{A} = X e^{D} X^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} e & 0 \\ 0 & 1/e \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} e & e \\ 0 & -1/e \end{pmatrix} = \boxed{\begin{pmatrix} e & e - 1/e \\ 0 & 1/e \end{pmatrix} = e^{A}}$$

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11. $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$A^2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2I$

$A^3 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = 2A = \begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix}$

$A^4 = 2A \cdot A = 2A^2 = 4I$

$A^5 = 4A$

$A^6 = 8I$

$A^7 = 8A \ \ldots$ etc.

$A^{2k} = 2^k I \quad k = 0, 1, 2, \ldots$

$A^{2k+1} = 2^k A \quad k = 0, 1, 2, \ldots$

$$e^{A} = \sum_{k=0}^{\infty} \frac{1}{2k!} 2^k I + \sum_{k=0}^{\infty} \frac{1}{2k+1!} 2^k A = \sum_{k=0}^{\infty} \frac{1}{2k!} 2^k I + A \sum_{k=0}^{\infty} \frac{1}{2k+1!} 2^k I$$

Since $a_{11} = a_{12} = a_{21} = 1$, $(e^{A})_{11,12,21} = \sum_{n=0}^{\infty} \frac{2^n}{n!}$, $\quad (e^{A})_{22} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}$