Name:________________________________
M511: Linear Algebra
Summer 2018
Comprehensive Final Exam (part I)

Instructions. *Complete all problems below, showing enough work. Read carefully and follow all instructions. You may not use any notes or electronic devices. All you need is a pencil and your brain.*

---

1. True/False [20 points] *Neatly write* **T** *on the line if the statement is always true, and* **F** *otherwise [1 point each]. In the space provided below the statement, give sufficient explanation of your answer [3 point each].*

_______**1.a.** Let $A, B, C \in \mathbb{R}^{n\times n}$. If A is similar to B and B is similar to C, then A is similar to C.

_______**1.b.** Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and $A \in \mathbb{R}^{n\times n}$ be a matrix representation of L with respect to any basis. If $L^k(\mathbf{x}) := L(L(\cdots L(\mathbf{x})\cdots))$, k-times, then A^k is a matrix representation of L^k with respect to the same basis.

_______**1.c.** If $\mathbf{x}$ and $\mathbf{y}$ are nonzero vectors in $\mathbb{R}^n$, then $\text{proj}_{\mathbf{y}}\mathbf{x} = \text{proj}_{\mathbf{x}}\mathbf{y}$.

_______**1.d.** Recall that a matrix $Q \in \mathbb{R}^{n\times n}$ is said to be *orthogonal* if and only if $Q^T = Q^{-1}$. If $Q_1, Q_2 \in \mathbb{R}^{n\times n}$ are orthogonal matrices, then Q_1Q_2 is also an orthogonal matrix.

_______**1.e.** If A and B are $n \times n$ matrices with the same eigenvalues, then they are similar matrices.

2. Find the best least squares fit line to the data.

x	-2	-1	0	1	2
y	0	1	3	5	6

3. Consider the matrix

$$A = \begin{pmatrix} 4 & -5 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

If possible, write A as a product XDX^{-1}, where D is a diagonal matrix and X is non-singular. If this is not possible, explain why.

4. Consider the linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}.$$

Find the kernel and image of L.

5. Consider the space $\mathbb{P}_3$ with inner product

$$\langle p, q\rangle = p(1)q(1) + p(2)q(2) + p(4)q(4).$$

Find the matrix representation of this inner product with respect to the standard basis of $\mathbb{P}_3$.

Name:_______________________________

M511: Linear Algebra

Summer 2018

Comprehensive Final Exam (part II)

Instructions. *Complete all problems below, showing enough work. Read carefully and follow all instructions. You may not use any notes or electronic devices. All you need is a pencil and your brain.*

---

6. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Compute the matrix exponential e^A.

7. Solve the initial value problem using your favorite method.

$$\begin{cases} y_1' = y_1 - 2y_2, \\ y_2' = 2y_2; \\ y_1(0) = 1, \\ y_2(0) = -3. \end{cases}$$

8. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ be nonzero vectors, and let θ be the smallest positive angle between them. Prove the following formula.

$$\mathbf{x}^T \mathbf{y} = \|\mathbf{x}\| \, \|\mathbf{y}\| \, \cos\theta$$

9. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Apply the Gram-Schmidt algorithm to find an orthonormal basis for the column space of A.

10. Consider the vector space $S = \text{span}\left\{e^{2t}\cos(t), e^{2t}\sin(t)\right\}$. Find the matrix representation of the derivative transformation on S,

$$D(f) = f'(t),$$

then use this matrix and the Fundamental Theorem of Calculus to compute

$$\int e^{2t}(3\cos t - 2\sin t)\,dt.$$