Name: Key

M511: Linear Algebra

Summer 2018

Comprehensive Final Exam (part I)

Instructions. *Complete all problems below, showing enough work. Read carefully and follow all instructions. You may not use any notes or electronic devices. All you need is a pencil and your brain.*

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1. True/False [20 points] *Neatly write* **T** *on the line if the statement is always true, and* **F** *otherwise [1 point each]. In the space provided below the statement, give sufficient explanation of your answer [3 point each].*

T **1.a.** Let $A, B, C \in \mathbb{R}^{n\times n}$. If A is similar to B and B is similar to C, then A is similar to C.

$A = SBS^{-1}$ and $B = TCT^{-1}$, so

$A = STCT^{-1}S^{-1}$ where $T^{-1}S^{-1} = (ST)^{-1}$.

T **1.b.** Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and $A \in \mathbb{R}^{n\times n}$ be a matrix representation of L with respect to any basis. If $L^k(\mathbf{x}) := L(L(\cdots L(\mathbf{x})\cdots))$, k-times, then A^k is a matrix representation of L^k with respect to the same basis.

$L\bar{x} = A\bar{x} \Rightarrow L^k\bar{x} = A(A(\cdots A\bar{x})\cdots)) = A^k\bar{x}$

F **1.c.** If $\mathbf{x}$ and $\mathbf{y}$ are nonzero vectors in $\mathbb{R}^n$, then $\text{proj}_{\mathbf{y}}\mathbf{x} = \text{proj}_{\mathbf{x}}\mathbf{y}$.

$\text{proj}_{\bar{y}}\bar{x} = \frac{\bar{x}^T\bar{y}}{\bar{y}^T\bar{y}}\bar{y}$ $\quad$ $\text{proj}_{\bar{x}}\bar{y} = \frac{\bar{x}^T\bar{y}}{\bar{x}^T\bar{x}}\bar{x}$

T **1.d.** Recall that a matrix $Q \in \mathbb{R}^{n\times n}$ is said to be *orthogonal* if and only if $Q^T = Q^{-1}$. If $Q_1, Q_2 \in \mathbb{R}^{n\times n}$ are orthogonal matrices, then Q_1Q_2 is also an orthogonal matrix.

$(Q_1Q_2)^T = Q_2^TQ_1^T = Q_2^{-1}Q_1^{-1} = (Q_1Q_2)^{-1}$

F **1.e.** If A and B are $n \times n$ matrices with the same eigenvalues, then they are similar matrices.

if $A \sim B$, then A and B have the same e'values, but not conversely.

2. Find the best least squares fit line to the data.

x	-2	-1	0	1	2
y	0	1	3	5	6

$xm + b = y$

$-2m + 1b = 0$
$-1m + 1b = 1$
$0m + 1b = 3$
$1m + 1b = 5$
$2m + 1b = 6$

$A = \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}$

$A^T = \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$

$\bar{y} = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 5 \\ 6 \end{pmatrix}$

$A^TA = \begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix}$

$(A^TA)^{-1} = \frac{1}{50}\begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix}$

$A^T\bar{y} = \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 3 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 16 \\ 15 \end{pmatrix}$

$\hat{x} = (A^TA)^{-1}A^T\bar{y} = \frac{1}{50}\begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix}\begin{pmatrix} 16 \\ 15 \end{pmatrix} = \frac{1}{50}\begin{pmatrix} 80 \\ 150 \end{pmatrix}$

The line is thus: $\boxed{y = \frac{8}{5}x + 3}$ or $\boxed{y = 1.6x + 3}$

3. Consider the matrix

$$A = \begin{pmatrix} 4 & -5 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

If possible, write A as a product XDX^{-1}, where D is a diagonal matrix and X is non-singular. If this is not possible, explain why.

$$|A-\lambda I| = (4-\lambda)\begin{vmatrix} -\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} - 1\begin{vmatrix} -5 & 1 \\ 1 & -1-\lambda \end{vmatrix} = (4-\lambda)\left[-\lambda(-1-\lambda)+1\right] - 1\left[-5(-1-\lambda)-1\right]$$

$$= (4-\lambda)(\lambda^2+\lambda+1) - (5\lambda+4)$$

$$= 4\lambda^2+4\lambda+\cancel{4}-\lambda^3-\lambda^2-\lambda-5\lambda-\cancel{4}$$

$$= -\lambda^3+3\lambda^2-2\lambda = -\lambda(\lambda^2-3\lambda+2) = -\lambda(\lambda-2)(\lambda-1)$$

$\boxed{\lambda = 0, 1, 2}$.

$\lambda_1 = 0$: $\left(\begin{array}{ccc|c} 4 & -5 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -5 & 5 & 0 \end{array}\right)$ $x_1 = t$, $x_2 = t$, $x_3 = t$ $\quad \bar{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 1$: $\left(\begin{array}{ccc|c} 3 & -5 & 1 & 0 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -2 & 4 & 0 \end{array}\right)$ $x_1 = 3t$, $x_2 = 2t$, $x_3 = t$ $\quad \bar{x}_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

$$X = \begin{pmatrix} 1 & 3 & 7 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$$

$\lambda_3 = 2$: $\left(\begin{array}{ccc|c} 2 & -5 & 1 & 0 \\ 1 & -2 & -1 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -2 & -1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & -1 & 3 & 0 \end{array}\right)$ $x_1 = 7t$, $x_2 = 3t$, $x_3 = t$ $\quad \bar{x}_3 = \begin{pmatrix} 7 \\ 3 \\ 1 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & 1 & -1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & -1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & -1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & -\frac{5}{2} & 7 & -\frac{7}{2} \\ 0 & 1 & 0 & -1 & 3 & -2 \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -2 & \frac{5}{2} \\ 0 & 1 & 0 & -1 & 3 & -2 \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right)$$

So $\boxed{A = \begin{pmatrix} 1 & 3 & 7 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -2 & \frac{5}{2} \\ -1 & 3 & -2 \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix}}$

4. Consider the linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$L\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ x_1 + x_2 \\ x_1 - x_2 \end{pmatrix}.$$

Find the kernel and image of L.

$A = \begin{pmatrix} -1 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$

$\mathrm{Ker}(L) = \mathrm{Null}(A) = \{\vec{0}\}$

$\mathrm{Image}(L) = \mathrm{Col}(A) = \mathrm{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}.$

5. Consider the space $\mathbb{P}_3$ with inner product

$$\langle p, q \rangle = p(1)q(1) + p(2)q(2) + p(4)q(4).$$

Find the matrix representation of this inner product with respect to the standard basis of $\mathbb{P}_3$.

$\langle 1,1 \rangle = 1^2 + 1^2 + 1^2 = 3$

$\langle 1,x \rangle = 1^2 + 1\cdot 2 \cdot 1\cdot 4 = 7$ $\quad \langle x,x \rangle = 1^2 + 2^2 + 4^2 = 1 + 4 + 16 = 21$

$\langle 1,x^2 \rangle = 1^2 + 1\cdot 4 + 1\cdot 16 = 21$ $\quad \langle x,x^2 \rangle = 1^2 + 2\cdot 2^2 + 4\cdot 4^2 = 1 + 8 + 64 = 73$ $\quad \langle x^2,x^2 \rangle = 1 + 2^4 + 4^4 = 1 + 16 + 256 = 273$

So,

$$\boxed{G = \begin{pmatrix} 3 & 7 & 21 \\ 7 & 21 & 73 \\ 21 & 73 & 273 \end{pmatrix}}$$

Name:________________________________
M511: Linear Algebra
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Wichita State University

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6. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Compute the matrix exponential e^A.

$$A^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

So, $$e^A = I + A + \frac{1}{2}A^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = e^A}$$

7. Solve the initial value problem using your favorite method.

$$\begin{cases} y_1' = y_1 - 2y_2, \\ y_2' = 2y_2; \\ y_1(0) = 1, \\ y_2(0) = -3. \end{cases}$$

$Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ $\quad A = \begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix}$ $\quad Y_0 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

$\lambda = 1, 2$

$\bar{x}_1 : \left(\begin{array}{cc|c} 0 & -2 & 0 \\ 0 & 1 & 0 \end{array}\right)$ $\quad \bar{x}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad \bar{x}_2 : \left(\begin{array}{cc|c} -1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right)$ $\quad \bar{x}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ $\qquad X = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$

$X^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

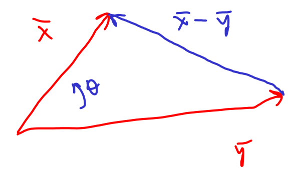
$X^{-1}Y_0 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -5 \\ -3 \end{pmatrix}$

so $\quad Y = Xe^{tD}(X^{-1}Y_0) = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} -5 \\ -3 \end{pmatrix} = \begin{pmatrix} e^t & -2e^{2t} \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} -5 \\ -3 \end{pmatrix} = \begin{pmatrix} -5e^t + 6e^{2t} \\ -3e^{2t} \end{pmatrix}$

and $\quad \boxed{\begin{array}{l} y_1(t) = 7e^t - 6e^{2t} \\ y_2(t) = -3e^{2t} \end{array}}$

8. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ be nonzero vectors, and let θ be the smallest positive angle between them. Prove the following formula.

$$\mathbf{x}^T \mathbf{y} = \|\mathbf{x}\| \, \|\mathbf{y}\| \cos\theta$$



The Law of Cosines states

$$\|\bar{x} - \bar{y}\|^2 = \|\bar{x}\|^2 + \|\bar{y}\|^2 - 2\|\bar{x}\| \|\bar{y}\| \cos\theta$$

$$\cancel{\|\bar{x}\|^2} \cancel{-2} \langle \bar{x}, \bar{y} \rangle + \cancel{\|\bar{y}\|^2} = \cancel{\|\bar{x}\|^2} + \cancel{\|\bar{y}\|^2} \cancel{-2} \|\bar{x}\| \|\bar{y}\| \cos\theta$$

After careful cancellations,

$$\langle \bar{x}, \bar{y} \rangle = \|\bar{x}\| \|\bar{y}\| \cos\theta \quad \checkmark$$

9. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Apply the Gram-Schmidt algorithm to find an orthonormal basis for the column space of A.

1. $\bar{r}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

2. $\|\bar{r}_1\| = \sqrt{4} = 2$

so, $\boxed{\bar{u}_1 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}$

3. $\bar{p}_2 = \langle \bar{a}_2, \bar{u}_1 \rangle \bar{u}_1 = \frac{1}{4}(2)\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

4. $\bar{r}_2 = \bar{a}_2 - \bar{p}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} \sim \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$

5. $\|\bar{r}_2\| = 2$

so, $\boxed{\bar{u}_2 = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}}$

6. $\bar{p}_3 = \langle \bar{a}_3, \bar{u}_1 \rangle \bar{u}_1 + \langle \bar{a}_3, \bar{u}_2 \rangle \bar{u}_2$

$= \frac{1}{4}(2)\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{4}(0)\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

7. $\bar{r}_3 = \bar{a}_3 - \bar{p}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \sim \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$

8. $\|\bar{r}_3\| = 2$

$\boxed{\bar{u}_3 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}}$

10. Consider the vector space $S = \text{span}\{e^{2t}\cos(t), e^{2t}\sin(t)\}$. Find the matrix representation of the derivative transformation on S,

$$D(f) = f'(t),$$

then use this matrix and the Fundamental Theorem of Calculus to compute

$$\int e^{2t}(3\cos t - 2\sin t)\,dt.$$

$D(e^{2t}\cos t) = 2e^{2t}\cos t - e^{2t}\sin t$

$D(e^{2t}\sin t) = e^{2t}\cos t + 2e^{2t}\sin t$

so $D = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$

$J = D^{-1} = \frac{1}{5}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$

so $\left[\int e^{2t}(3\cos t - 2\sin t)\,dt\right] = J\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 8 \\ -1 \end{pmatrix}$

and

$$\boxed{\int e^{2t}(3\cos t - 2\sin t)\,dt = \frac{8}{5}e^{2t}\cos t - \frac{1}{5}e^{2t}\sin t + C}$$