Name: Key

M511: Linear Algebra (Summer 2018)

Good Problems 1: Chapter 1

Instructions *Complete all problems on this paper, showing enough work. A selection of problems will be graded based on the organization and clarity of the work shown in addition to the final solution (provided one exists).*

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1. Solve the system of equations by using Gaussian or Gauss-Jordan elimination.

$$\begin{cases} x_2 + x_3 + x_4 & = 0 \\ 3x_1 + 3x_3 - 4x_4 & = 7 \\ x_1 + x_2 + x_3 + 2x_4 & = 6 \\ 2x_1 + 3x_2 + x_3 + 3x_4 & = 6 \end{cases}$$

$$\left(\begin{array}{cccc|c} 0 & 1 & 1 & 1 & 0 \\ 3 & 0 & 3 & -4 & 7 \\ 1 & 1 & 1 & 2 & 6 \\ 2 & 3 & 1 & 3 & 6 \end{array}\right)$$

$R_1 \leftrightarrow R_3$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 6 \\ 3 & 0 & 3 & -4 & 7 \\ 0 & 1 & 1 & 1 & 0 \\ 2 & 3 & 1 & 3 & 6 \end{array}\right)$$

$R_2 - 3R_1 \to R_2$
$R_4 - 2R_1 \to R_4$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 6 \\ 0 & -3 & 0 & -10 & -11 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & -1 & -6 \end{array}\right)$$

$R_2 \leftrightarrow R_3$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 6 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & -3 & 0 & -10 & -11 \\ 0 & 1 & -1 & -1 & -6 \end{array}\right)$$

$R_3 + 3R_2 \to R_3$
$-R_4 + R_2 \to R_4$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 6 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 3 & -7 & -11 \\ 0 & 0 & 2 & 2 & 6 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 6 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 3 & -7 & -11 \\ 0 & 0 & 2 & 2 & 6 \end{array}\right)$$

$\frac{1}{2}R_4 \to R_4$, then $R_3 - 3R_4 \to R_3$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 6 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -10 & -20 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right)$$

$-\frac{1}{10}R_3 \to R_3$, then $R_3 \leftrightarrow R_4$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 2 & 6 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right)$$

$x_1 + x_2 + x_3 + 2x_4 = 6 \Rightarrow x_1 = 4$

$x_2 + x_3 + x_4 = 0 \Rightarrow x_2 = -3$

$x_3 + x_4 = 3 \Rightarrow x_3 = 1$

$x_4 = 2$

So $\bar{x} = (4, -3, 1, 2)$ is the solution

2. Let A be a matrix of the form

$$A = \begin{pmatrix} \alpha & \beta \\ 2\alpha & 2\beta \end{pmatrix}$$

where α and β are fixed constants not both equal to 0.

a.) Explain why the system $A\mathbf{x} = (3,\ 1)^T$ must be inconsistent.

b.) how can one choose a nonzero vector $\mathbf{b}$ so that the system $A\mathbf{x} = \mathbf{b}$ will be consistent? Explain.

a.) $\left(\begin{array}{cc|c} \alpha & \beta & 3 \\ 2\alpha & 2\beta & 1 \end{array}\right)$

$R_2 - 2R_1 \to R_2$

$\left(\begin{array}{cc|c} \alpha & \beta & 3 \\ 0 & 0 & -5 \end{array}\right)$

This line is inconsistent $(0=-5)$ so the system is inconsistent.

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b.) $\left(\begin{array}{cc|c} \alpha & \beta & b_1 \\ 2\alpha & 2\beta & b_2 \end{array}\right) \xrightarrow{R_2 - 2R_1 \to R_2} \left(\begin{array}{cc|c} \alpha & \beta & b_1 \\ 0 & 0 & b_2 - 2b_1 \end{array}\right)$

For this system to be consistent $b_2 - 2b_1 = 0$ must hold

so, $\bar{b} = \gamma \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ for any $\gamma \in \mathbb{R}$.

3. Let A be a 3×3 matrix and let $\mathbf{b} = 3\mathbf{a}_1 + \mathbf{a}_2 + 4\mathbf{a}_3$. Will the system $A\mathbf{x} = \mathbf{b}$ be consistent? Explain.

$$A\bar{x} = (\bar{a}_1, \bar{a}_2, \bar{a}_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1\bar{a}_1 + x_2\bar{a}_2 + x_3\bar{a}_3.$$

Since $\bar{b} = 3\bar{a}_1 + \bar{a}_2 + 4\bar{a}_3$, then $\bar{x} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ is a solution of $A\bar{x} = \bar{b}$.

Therefore the system has at least one solution and is <u>consistent</u>.

4. Let A be a 3×3 matrix and suppose that $\mathbf{a}_1 - 3\mathbf{a}_2 + 2\mathbf{a}_3 = \mathbf{0}$. Is A nonsingular? Explain.

By similar reasoning as to that above,

$\bar{x} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$ is a solution of the equation $A\bar{x} = \bar{0}$.

If A^{-1} existed, then $\bar{x} = A^{-1}\bar{0} = \bar{0}$ must hold. But $\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Therefore A must be singular.

In general, by Theorem 1.5.2, since $x = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$ is a nontrivial solution of the equation $A\bar{x} = \bar{0}$, then A is singular.

5. Let E and F be $n \times n$ elementary matrices and let $C = EF$. Is C nonsingular? Explain.

If C is nonsingular, then

$$CC^{-1} = I$$
$$EF\,C^{-1} = I$$
$$F\,C^{-1} = E^{-1}I = E^{-1}$$
$$\boxed{C^{-1} = F^{-1}E^{-1}}$$

Since E and F are elementary matrices, then E^{-1} and F^{-1} exist and the formula

$$C^{-1} = F^{-1}E^{-1}$$

holds true.

6. Let A and B be 10×10 matrices that are partitioned into submatrices

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

a.) If $A_{11} \in \mathbb{R}^{6\times 5}$ and $B_{11} \in \mathbb{R}^{k\times r}$, what conditions, if any, must k and r satisfy in order to make the block multiplication of A times B possible?

b.) Assuming that the block multiplication is possible, how would the $(2,2)$ block of the product be determined?

a.) $A_{11} \in \mathbb{R}^{6\times 5}$ implies $B_{11} \in \mathbb{R}^{5\times r}$ where $1 \le r \le 9$.

b.) $(AB)_{22} = A_{21}B_{12} + A_{22}B_{22}$