Name: Key

M511: Linear Algebra (Summer 2018)
Good Problems 3: Sections 3.1–3.4

WICHITA STATE UNIVERSITY

Instructions *Complete all problems on this paper, showing enough work. A selection of problems will be graded based on the organization and clarity of the work shown in addition to the final solution (provided one exists).*

---

1. In $\mathbb{R}^3$ let $\mathbf{x}_1$ and $\mathbf{x}_2$ be linearly independent vectors and let $\mathbf{x}_3 = \mathbf{0}$. Are $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ linearly independent? Prove your answer.

No, $\{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$ are linearly dependent since

$$0\bar{x}_1 + 0\bar{x}_2 + 1\bar{x}_3 = \bar{0} + \bar{0} + \bar{0} = \bar{0}.$$

2. Consider the subset $S = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1 \cdot x_2 = 0\}$. Is S a subspace of $\mathbb{R}^2$? Prove your answer.

S1. Let $\alpha \in \mathbb{R}$, $\alpha\bar{x} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix}$. Then $(\alpha x_1)\cdot(\alpha x_2) = \alpha^2(x_1 \cdot x_2) = \alpha^2(0) = 0$.

S2. Let $\bar{x}, \bar{y} \in S$, $\bar{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\bar{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. $\bar{x} + \bar{y} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$

$$(x_1 + y_1)(x_2 + y_2) = \underbrace{x_1 x_2}_{=0} + x_1 y_2 + y_1 x_2 + \underbrace{y_1 y_2}_{=0} = x_1 y_2 + x_2 y_1$$

if $x_1 = y_2 = 0$ but $x_2, y_1 \neq 0$, then $\bar{x}, \bar{y} \in S$, but S2 fails.

Hence this S is not a subspace of $\mathbb{R}^2$. ∎

3. Consider the vectors

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 1 \\ 5 \\ 5 \end{pmatrix}, \quad \mathbf{x}_4 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

a.) Are $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, and $\mathbf{x}_4$ linearly independent in $\mathbb{R}^3$? Explain.

b.) Do $\mathbf{x}_1, \mathbf{x}_2$ span $\mathbb{R}^3$? Explain.

c.) Do $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ span $\mathbb{R}^3$? Are they linearly independent? Do they form a basis for $\mathbb{R}^3$? Explain.

d.) Do $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_4$ span $\mathbb{R}^3$? Are they linearly independent? Do they form a basis for $\mathbb{R}^3$? Explain.

a.) No, 4 vectors in $\mathbb{R}^3$ can never be linearly independent.

b.) No, 2 vectors cannot span $\mathbb{R}^3$.

c.) $\{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$ are linearly dependent since rows 2 and 3 of the matrix $(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ are the same. Hence, they cannot form a basis of $\mathbb{R}^3$.

d.) $(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 2 & 3 & 3 \end{pmatrix}$ has determinant $\begin{vmatrix} 3 & 2 \\ 3 & 3 \end{vmatrix} - \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} = (9-6) - (6-4) - 0 = 3 - 2 = 1 \neq 0$.

Therefore the vectors $\{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$ are linearly independent. Since three l.i. vectors in $\mathbb{R}^3$ span $\mathbb{R}^3$, then $\{\bar{x}_1, \bar{x}_2, \bar{x}_3\}$ form a basis for $\mathbb{R}^3$. ∎

4. Let S be the set of all symmetric 2×2 matrices with real entries. (*a*) Show that S is a subspace of $\mathbb{R}^{2\times 2}$; (*b*) Find a basis for S.

$S = \left\{ \begin{pmatrix} a & b \\ b & d \end{pmatrix} \in \mathbb{R}^{2\times 2} \,\middle|\, a,b,d \in \mathbb{R} \right\} = \left\{ A \in \mathbb{R}^{2\times 2} \,\middle|\, a_{ij} = a_{ji} \text{ for all } i,j = 1,2 \right\}$

a.) S1: $A \in S$, αA satisfies $(\alpha A)_{ij} = \alpha a_{ij}$.

Since $a_{ij} = a_{ji}$, then $\alpha a_{ij} = \alpha a_{ji}$. ■

S2: $A, B \in S$, $(A+B)_{ij} = a_{ij} + b_{ij} = a_{ji} + b_{ji} = (A+B)_{ji}$.

Therefore S is a subspace of $\mathbb{R}^{2\times 2}$.

---

b.) $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ form a basis for S.

They are linearly independent since

$c_1 A_1 + c_2 A_2 + c_3 A_3 = \begin{pmatrix} c_1 & c_3 \\ c_3 & c_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ if and only if $c_1 = c_2 = c_3 = 0$.

They span S since for any $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix} \in S$,

$A = aA_1 + bA_2 + dA_3$. ■

5. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ be linearly independent vectors in $\mathbb{R}^4$ and let $A \in \mathbb{R}^{4\times 4}$ be nonsingular. Prove that if $\mathbf{y}_1 = A\mathbf{x}_1$, $\mathbf{y}_2 = A\mathbf{x}_2$, $\mathbf{y}_3 = A\mathbf{x}_3$, then $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ are linearly independent.

$$\begin{aligned} c_1\bar{y}_1 + c_2\bar{y}_2 + c_3\bar{y}_3 &= c_1 A\bar{x}_1 + c_2 A\bar{x}_2 + c_3 A\bar{x}_3 \\ &= A(c_1\bar{x}_1 + c_2\bar{x}_2 + c_3\bar{x}_3) = \bar{0} \end{aligned}$$

Since A is nonsingular, then $c_1\bar{x}_1 + c_2\bar{x}_2 + c_3\bar{x}_3 = \bar{0}$ (by thm 1.5.2.).

Since $\bar{x}_1, \bar{x}_2, \bar{x}_3$ are linearly independent, then $c_1 = c_2 = c_3 = 0$.

Therefore $c_1\bar{y}_1 + c_2\bar{y}_2 + c_3\bar{y}_3 = \bar{0}$ implies $c_1 = c_2 = c_3 = 0$, hence

$\bar{y}_1, \bar{y}_2, \bar{y}_3$ are linearly independent. ■