Name:________________________________

M511: Linear Algebra (Summer 2018)

Good Problems 4: Sections 3.5, 3.6, 4.1–4.3

WICHITA STATE UNIVERSITY

Instructions *Complete all problems on this paper, showing enough work. A selection of problems will be graded based on the organization and clarity of the work shown in addition to the final solution (provided one exists).*

---

1. Let V, W be finite dimensional vector spaces, and let $\mathscr{L}(V, W)$ denote the set of all linear transformations from V to W. Prove that $\mathscr{L}(V, W)$ is itself a vector space; *i.e.,* verify the 10 vector space axioms.

2. Let V be a vector space with basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$. The *dual space* of V is the vector space $V^* = \mathscr{L}(V, \mathbb{R})$ of linear transformations from V to $\mathbb{R}$. Prove that $\{\omega_1, \ldots, \omega_n\}$ form a basis for V^*, where

$$\omega_i(\mathbf{x}_j) = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

3. How do the dimensions of the null space and column space of a matrix relate to the number of lead and free variables in the reduced row echelon form of the matrix? Explain.

4. Let $A \in \mathbb{R}^{6\times 4}$ with $\text{rk}(A) = 4$. (a) What is the dimension of $\text{Null}(A)$? (b) What is the dimension of $\text{Col}(A)$? (c) If $\mathbf{b} \in \text{Col}(A)$, how many solutions will the linear system $A\mathbf{x} = \mathbf{b}$ have?

5. Consider the vector space $V = \text{span}\{e^t, te^t\}$. (*a*) Find the matrix representations of the linear transformations $D : V \to V$ and $J : V \to V$. (*b*) Use the matrix representation of J to compute $\int (3-4t)e^t \, dt$.

$$D(f) = f'(t), \quad J(f) = \int f(t)\, dt.$$