**Name**: Key

**M511: Linear Algebra** (Summer 2018)
Good Problems 4: Sections 3.5, 3.6, 4.1–4.3

---

**Instructions** *Complete all problems on this paper, showing enough work. A selection of problems will be graded based on the organization and clarity of the work shown in addition to the final solution (provided one exists).*

---

**1.** Let $V, W$ be finite dimensional vector spaces, and let $\mathscr{L}(V, W)$ denote the set of all linear transformations from $V$ to $W$. Prove that $\mathscr{L}(V, W)$ is itself a vector space; *i.e.*, verify the 10 vector space axioms.

Operations: $(\alpha \cdot L)(\vec{x}) := \alpha \otimes (L(\vec{x}))$ where $\otimes$ is the scalar mult. in $W$.

$(L_1 + L_2)(\vec{x}) := L_1(\vec{x}) \oplus L_2(\vec{x})$ where $\oplus$ is the vector add. in $W$,

All 10 axioms should be straight forward. All of the A's hold for $\mathcal{L}(V,W)$ since $L$ is linear (L1 and L2) and the A's hold in $W$.

(write 'em out!)

**2.** Let $V$ be a vector space with basis $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$. The *dual space* of $V$ is the vector space $V^* = \mathscr{L}(V, \mathbb{R})$ of linear transformations from $V$ to $\mathbb{R}$. Prove that $\{\omega_1, \ldots, \omega_n\}$ form a basis for $V^*$, where

$$\omega_i(\mathbf{x}_j) = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

<u>Spanning</u>: A linear transformation $L \in \mathcal{L}(V, \mathbb{R})$ satisfies

$$L(\bar{x}) = L(c_1\bar{x}_1 + c_2\bar{x}_2 + \cdots + c_n\bar{x}_n) = c_1 L(\bar{x}_1) + c_2 L(\bar{x}_2) + \cdots + c_n L(\bar{x}_n)$$

$$= \omega_1(c_1\bar{x}_1 + c_2\bar{x}_2 + \cdots + c_n\bar{x}_n) \cdot L(\bar{x}_1) + \cdots$$

$$= \sum_{k=1}^{n} \omega_k(c_1\bar{x}_1 + c_2\bar{x}_2 + \cdots + c_n\bar{x}_n) \cdot L(\bar{x}_k)$$

$$= \sum_{k=1}^{n} a_k \omega_k(\bar{x}) = \left(\sum_{k=1}^{n} a_k \omega_k\right)(\bar{x})$$

Since $\omega_k(c_1\bar{x}_1 + \cdots + c_n\bar{x}_n) = \omega_k(c_1\bar{x}_1) + \cdots + \omega_k(c_n\bar{x}_n)$
$= c_1\omega_k(\bar{x}_1) + \cdots + c_n\omega_k(\bar{x}_n)$
$= c_k\omega_k(\bar{x}_k)$
$= c_k$. for all $k$.

Since $L: V \to \mathbb{R}$, $L(\bar{x}_k) \in \mathbb{R}$ for each basis vector $\bar{x}_k$. Call this number $a_k$.

This implies that $L = \sum_{k=1}^{n} a_k\omega_k$ for some coefficients $a_1, \ldots, a_n$.
Therefore $\omega_1, \ldots, \omega_n$ span $\mathcal{L}(V, \mathbb{R})$.

<u>Linear independence</u>: The $\bar{0}$-transformation satisfies $\bar{0}(\bar{x}_k) = 0$ for all $\bar{x}_k$, so each $a_k$ in the above argument must equal 0. Thus,

$$\bar{0} = 0\omega_1 + 0\omega_2 + \cdots + 0\omega_n$$

is unique. ∎

**3.** How do the dimensions of the null space and column space of a matrix relate to the number of lead and free variables in the reduced row echelon form of the matrix? Explain.

The number of lead variables is the dimension of the col. space. This is because the columns corresponding to the lead variables span the column space and are l.i.

The number of free variables is the dimension of the null space. The free variables are the independent variables in the solution of the homog. eq'n, hence the dimension of the null space.

**4.** Let $A \in \mathbb{R}^{6\times4}$ with $\mathrm{rk}(A) = 4$. (*a*) What is the dimension of $\mathrm{Null}(A)$? (*b*) What is the dimension of $\mathrm{Col}(A)$? (*c*) If $\mathbf{b} \in \mathrm{Col}(A)$, how many solutions will the linear system $A\mathbf{x} = \mathbf{b}$ have?

a) $\mathrm{nullity}(A) = 0$ since A has 4 columns and $\mathrm{rk}(A) = \dim(\mathrm{Col}(A)) = 4$.

b) $\dim(\mathrm{Col}(A)) = 4$ by def'n of rank.

c) $A\bar{x} = \bar{b}$ will have exactly 1 solution.

**5.** Consider the vector space $V = \text{span}\{e^t, te^t\}$. ($a$) Find the matrix representations of the linear transformations $D: V \to V$ and $J: V \to V$. ($b$) Use the matrix representation of $J$ to compute $\int (3-4t)e^t\,dt$.

$$D(f) = f'(t), \quad J(f) = \int f(t)\,dt.$$

$D(e^t) = e^t \rightsquigarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$D(te^t) = (1+t)e^t \rightsquigarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

so $\boxed{D = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}$

Then $\boxed{J = D^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}}$

and $\int (3-4t)e^t\,dt \approx \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 7 \\ -4 \end{pmatrix} \rightsquigarrow (7-4t)e^t + C$

Not nec. in this class.