Name: Key
M511: Linear Algebra (Summer 2018)
Good Problems 5: Chapter 5

WICHITA STATE UNIVERSITY

Instructions *Complete all problems on this paper, showing enough work. A selection of problems will be graded based on the organization and clarity of the work shown in addition to the final solution (provided one exists).*

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1. Let

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -2 \\ 1 \\ 2 \\ 0 \end{pmatrix}.$$

a) Find the vector projection $\mathbf{p}$ of $\mathbf{x}$ onto $\mathbf{y}$.

b) Verify that $\mathbf{x} - \mathbf{p}$ is orthogonal to $\mathbf{p}$.

c) Verify that the Pythagorean Law holds for $\mathbf{x}, \mathbf{p}$, and $\mathbf{x} - \mathbf{p}$.

a) $\bar{p} = \frac{\bar{x}^T \bar{y}}{\bar{y}^T \bar{y}} \bar{y} = \frac{-2+1+4+0}{4+1+4+0} \begin{pmatrix} -2 \\ 1 \\ 2 \\ 0 \end{pmatrix} = \frac{3}{9} \begin{pmatrix} -2 \\ 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -2/3 \\ 1/3 \\ 2/3 \\ 0 \end{pmatrix}$

b) $\bar{x} - \bar{p} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix} - \begin{pmatrix} -2/3 \\ 1/3 \\ 2/3 \\ 0 \end{pmatrix} = \begin{pmatrix} 5/3 \\ 2/3 \\ 4/3 \\ 2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 5 \\ 2 \\ 4 \\ 6 \end{pmatrix}$

$(\bar{x} - \bar{p})^T \bar{p} = \frac{1}{9} (5, 2, 4, 6) \begin{pmatrix} -2 \\ 1 \\ 2 \\ 0 \end{pmatrix} = \frac{1}{9} (-10 + 2 + 8 + 0) = 0.$

c.) $\|\bar{x}\|^2 = \bar{x}^T \bar{x} = 1+1+4+4 = 10$

$\|\bar{p}\|^2 = \bar{p}^T \bar{p} = \frac{1}{9}(4+1+4) = 9/9 = 1$

$\|\bar{x} - \bar{p}\|^2 = (\bar{x} - \bar{p})^T (\bar{x} - \bar{p}) = \frac{1}{9}(25 + 4 + 16 + 36) = \frac{1}{9} \cdot 81 = 9$

Indeed $9 + 1 = 10$

$\|\bar{x} - \bar{p}\|^2 + \|\bar{p}\|^2 = \|\bar{x}\|^2$ ✓

2. Prove the *Triangle Inequality* in $\mathbb{R}^n$:

$$\|\mathbf{x}_1 + \mathbf{x}_2\| \le \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$$

for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$.

$$\begin{aligned}
\|\bar{x}_1+\bar{x}_2\|^2 &= \langle \bar{x}_1+\bar{x}_2, \bar{x}_1+\bar{x}_2\rangle \\
&= \langle \bar{x}_1,\bar{x}_1\rangle + 2\langle \bar{x}_1,\bar{x}_2\rangle + \langle \bar{x}_2,\bar{x}_2\rangle \\
&\le \|\bar{x}_1\|^2 + 2|\langle \bar{x}_1,\bar{x}_2\rangle| + \|\bar{x}_2\|^2 && \text{properties of abs. value.} \\
&\le \|\bar{x}_1\|^2 + 2\|x_1\|\|x_2\| + \|\bar{x}_2\|^2 && \text{CSB} \neq \\
&= (\|\bar{x}_1\| + \|\bar{x}_2\|)^2
\end{aligned}$$

since $\|\cdot\| \ge 0$ always, this implies

$$\|\bar{x}_1+\bar{x}_2\| \le \|\bar{x}_1\| + \|\bar{x}_2\|. \quad \checkmark$$

3. Prove the *Cauchy-Schwarz-Bunyachevsky Inequality* in $\mathbb{R}^n$:

$$|\mathbf{x}^T\mathbf{y}| \le \|\mathbf{x}\|\,\|\mathbf{y}\|$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

if either $\bar{x}=\bar{0}$ or $\bar{y}=\bar{0}$, then $\bar{x}^T\bar{y}=0$ and $\|\bar{x}\|\|\bar{y}\|=0$.

if neither $\bar{x}$ nor $\bar{y}=\bar{0}$, then let θ be the smallest $\measuredangle$ between them.

We have

$$\bar{x}^T\bar{y} = \|\bar{x}\|\|\bar{y}\|\cos\theta,$$

so

$$|\bar{x}^T\bar{y}| = \big|\|\bar{x}\|\|\bar{y}\|\cos\theta\big| = \|\bar{x}\|\|\bar{y}\||\cos\theta|$$

but $|\cos\theta| \le 1$, so

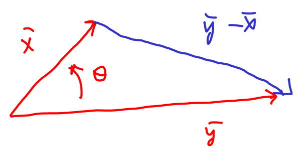
$$|\bar{x}^T\bar{y}| \le \|\bar{x}\|\|\bar{y}\|\cdot 1.$$

Equality holds only if $|\cos\theta|=1$, whence $\bar{x}\,/\!/\,\bar{y}$.

4. Use trigonometry in $\mathbb{R}^n$ to prove the formula

$$\mathbf{x}^T\mathbf{y} = \|\mathbf{x}\|\,\|\mathbf{y}\|\cos\theta$$

where θ is the smallest angle between $\mathbf{x}$ and $\mathbf{y}$. For $\bar{x}, \bar{y} \neq \bar{0}$!



By the Law of Cosines,

$$\|\bar{y}-\bar{x}\|^2 = \|\bar{y}\|^2 + \|\bar{x}\|^2 - 2\|\bar{x}\|\|\bar{y}\| \cos\theta$$

$$\cancel{\|\bar{y}\|^2} - 2\langle \bar{x},\bar{y}\rangle + \cancel{\|\bar{x}\|^2} = \cancel{\|\bar{y}\|^2} + \cancel{\|\bar{x}\|^2} - 2\|\bar{x}\|\|\bar{y}\| \cos\theta$$

so,

$$\langle \bar{x},\bar{y}\rangle = \bar{x}^T\bar{y} = \|\bar{x}\|\|\bar{y}\| \cos\theta .$$

5. Prove the *Parallelogram Law* in $\mathbb{R}^n$:

$$\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

$$\|\bar{x}+\bar{y}\|^2 = \|\bar{x}\|^2 + 2\langle \bar{x},\bar{y}\rangle + \|\bar{y}\|^2$$

$$+ \quad \|\bar{x}-\bar{y}\|^2 = \|\bar{x}\|^2 - 2\langle \bar{x},\bar{y}\rangle + \|\bar{y}\|^2$$

$$\|\bar{x}+\bar{y}\|^2 + \|\bar{x}-\bar{y}\|^2 = 2\|\bar{x}\|^2 + 2\|\bar{y}\|^2 .$$

6. Apply the Gram-Schmidt algorithm to find an orthonormal basis for the column space of the matrix

$$A = \begin{pmatrix} 1 & -3 & -5 \\ 1 & 1 & -2 \\ 1 & -3 & 1 \\ 1 & 1 & 4 \end{pmatrix}$$

1. $\bar{r}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

2. $\|\bar{r}_1\| = \sqrt{4} = 2$

$$\boxed{\bar{u}_1 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}$$

3. $\bar{p}_2 = \langle \bar{a}_2, \bar{u}_1 \rangle \bar{u}_1 = \frac{1}{2}(-3+1-3+1) \cdot \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{4}(-4)\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$

4. $\bar{r}_2 = \bar{a}_2 - \bar{p}_2 = \begin{pmatrix} -3 \\ 1 \\ -3 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ -2 \\ 2 \end{pmatrix} = 2\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \sim \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$

5. $\|\bar{r}_2\| = \sqrt{4} = 2$

$$\boxed{\bar{u}_2 = \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}}$$

6. $\bar{p}_3 = \langle \bar{a}_3, \bar{u}_1 \rangle \bar{u}_1 + \langle \bar{a}_3, \bar{u}_2 \rangle \bar{u}_2 = \frac{1}{4}(-5-2+1+4)\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{4}(5-2-1+4)\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{3}{2}\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -2 \\ 1 \end{pmatrix}$

7. $\bar{r}_3 = \bar{a}_3 - \bar{p}_3 = \begin{pmatrix} -5 \\ -2 \\ 1 \\ 4 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \\ 3 \\ 3 \end{pmatrix} \sim \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$

8. $\|\bar{r}_3\| = \sqrt{4} = 2$

$$\boxed{\bar{u}_3 = \frac{1}{2}\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}}$$

7. Find the best least squares fit line to the data.

x	-1	1	2
y	1	3	3

$xm + b = y$

$-1m + b = 1$

$1m + b = 3$

$2m + b = 3$

$A = \begin{pmatrix} -1 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}$ $\qquad \bar{b} = \begin{pmatrix} 1 \\ 3 \\ 3 \end{pmatrix}$

$A^T = \begin{pmatrix} -1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$

$A^T A = \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix}$ $\qquad A^T \bar{b} = \begin{pmatrix} 8 \\ 7 \end{pmatrix}$

$A^T A^{-1} = \frac{1}{14} \begin{pmatrix} 3 & -2 \\ -2 & 6 \end{pmatrix}$

$\hat{x} = (A^T A)^{-1} A^T \bar{b} = \frac{1}{14} \begin{pmatrix} 3 & -2 \\ -2 & 6 \end{pmatrix} \begin{pmatrix} 8 \\ 7 \end{pmatrix} = \frac{1}{14} \begin{pmatrix} 24 - 14 \\ -16 + 42 \end{pmatrix} = \frac{1}{14} \begin{pmatrix} 10 \\ 26 \end{pmatrix} = \begin{pmatrix} 5/7 \\ 13/7 \end{pmatrix}$

so the line is:

$\boxed{y = \frac{5}{7}x + \frac{13}{7}}$

8. Consider the inner product on $\mathbb{P}_n$ defined by

$$\langle p, q \rangle = \int_{-1}^{1} p(x)q(x)\,dx.$$

Starting with the standard basis $\{1, x, x^2, x^3\}$ of $\mathbb{P}_4$, use the Gram-Schmidt algorithm to find an orthonormal basis of $\mathbb{P}_4$ with respect to this inner product.

Solution is posted in slack!