Name: Key

M511: Linear Algebra (Summer 2018)

Good Problems 6: Chapter 6

WICHITA STATE UNIVERSITY

Instructions *Complete all problems on this paper, showing enough work. A selection of problems will be graded based on the organization and clarity of the work shown in addition to the final solution (provided one exists).*

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1. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 2 & -2 \end{pmatrix}.$$

a.) Find all eigenvalues and eigenspaces of A.

b.) Factor A into a product $A = XDX^{-1}$, where D is a diagonal matrix.

c.) Use your answer to part (*b*) to compute A^7.

a.) $|A-\lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 1-\lambda & -1 \\ 1 & 2 & -2-\lambda \end{vmatrix} = 1-\lambda \begin{vmatrix} 1-\lambda & -1 \\ 2 & -2-\lambda \end{vmatrix} = (1-\lambda)\Big((1-\lambda)(-2-\lambda)+2\Big) = -(\lambda-1)^2(\lambda+2) - 2\lambda + 2$

$= -(\lambda^2-2\lambda+1)(\lambda+2) - 2\lambda + 2$

$= -(\lambda^3 - \cancel{2\lambda^2} + \lambda + \cancel{2\lambda^2} - 4\lambda + 2) - 2\lambda + 2$

$= -\lambda^3 + \lambda = -\lambda(\lambda+1)(\lambda-1)$

so $\boxed{\lambda = 0, 1, -1}$

$(X|I) = \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right)$

$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & -1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right)$

$\underbrace{\hphantom{xxx}}_{X^{-1}}$

$\lambda_1 = 0$: $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & 2 & -2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $x_1 = 0$, $x_2 = t$, $x_3 = t$ $\quad \bar{x}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 1$: $\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 2 & -3 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $x_1 = t$, $x_2 = t$, $x_3 = t$ $\quad \bar{x}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\lambda_3 = -1$: $\left(\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 1 & 2 & -1 & 0 \\ 1 & 2 & -1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ $x_1 = 0$, $x_2 = \frac{1}{2}t$, $x_3 = t$ $\quad \bar{x}_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

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b). $A = XDX^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 & 2 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}$

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c.) $A^7 = XD^7X^{-1} = X \begin{pmatrix} 0^7 & 0 & 0 \\ 0 & 1^7 & 0 \\ 0 & 0 & -1 \end{pmatrix} X^{-1} = XDX^{-1} = A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 2 & -2 \end{pmatrix}$

2. Let A be a nonsingular $n \times n$ matrix, and let λ be an eigenvalue of A. *a.*) Show that $\lambda \neq 0$; *b.*) Show that $\dfrac{1}{\lambda}$ is an eigenvalue of A^{-1}.

a.) Let $\lambda = 0$ be an e'value of A. Then $|A - 0I| = |A| = 0$. But A is nonsingular, so this is a contradiction. Thus, $\lambda \neq 0$.

b.) Let λ be an e'value of A, and $|A| \neq 0$.

Then $A\bar{x} = \lambda \bar{x}$ for some $\bar{x} \neq \bar{0}$.

$\Rightarrow \bar{x} = A^{-1}(\lambda \bar{x})$

$\Rightarrow \bar{x} = \lambda A^{-1} \bar{x}$

$\Rightarrow \frac{1}{\lambda} \bar{x} = A^{-1} \bar{x}$

Therefore $\frac{1}{\lambda}$ is an eigenvalue for A^{-1}!

3. Let A be a matrix whose columns all add up to a fixed constant δ. Show that δ is an eigenvalue of A.

$A = (\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n)$ where $\bar{a}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{pmatrix}$ and $a_{1j} + a_{2j} + \cdots + a_{nj} = \delta$ for all $j = 1, \ldots, n$.

$\Rightarrow \left(\sum_{k=1}^{n} a_{kj} \right) - \delta = 0$ for all j

$\Rightarrow \sum_{j=1}^{n} \left(\sum_{k=1}^{n} (a_{kj} - (\delta I)_{kj}) \right) = 0$

$\Rightarrow |A - \delta I| = 0$.

Hence δ is an e'value of A.

4. Consider the matrices

$$A = \begin{pmatrix} 1 & -2 \\ 3 & -4 \end{pmatrix}, \quad \text{and} \quad \mathbf{Y}_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Compute e^{tA} and use it to compute the solution of the initial value problem $\mathbf{Y}'(t) = A\mathbf{Y}$, $\mathbf{Y}(0) = \mathbf{Y}_0$.

$p(\lambda) = \lambda^2 + 3\lambda + 2 = (\lambda+2)(\lambda+1) \quad \lambda_1 = -1 \quad \lambda_2 = -2$

$\lambda_1 = -1: \left(\begin{array}{cc|c} 2 & -2 & 0 \\ 3 & -3 & 0 \end{array}\right) \quad \bar{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \lambda_2 = -2: \left(\begin{array}{cc|c} 3 & -2 & 0 \\ 3 & -2 & 0 \end{array}\right) \quad \bar{x}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$X = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \quad X^{-1} = \frac{1}{1}\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \quad e^{tD} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}$

$$e^{tA} = Xe^{tD}X^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} e^{-t} & 2e^{-2t} \\ e^{-t} & 3e^{-2t} \end{pmatrix}\begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$$

$$\boxed{e^{tA} = \begin{pmatrix} 3e^{-t} - 2e^{-2t} & -2e^{-t} + 2e^{-2t} \\ 3e^{-t} - 3e^{-2t} & -2e^{-t} + 3e^{-2t} \end{pmatrix}}$$

so $Y = e^{tA} Y_0 = \begin{pmatrix} 3e^{-t} - 2e^{-2t} & -2e^{-t} + 2e^{-2t} \\ 3e^{-t} - 3e^{-2t} & -2e^{-t} + 3e^{-2t} \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\Rightarrow \boxed{\begin{cases} y_1 = -e^{-t} + 2e^{-2t} \\ y_2 = -e^{-t} + 3e^{-2t} \end{cases}}$$

5. Consider the real vector space $\mathbb{C} = \{z = x + iy \mid x, y \in \mathbb{R}\}$, and the linear transformation $L : \mathbb{C} \to \mathbb{C}$ defined by $L(z) = iz$. What do you think the eigenvalues and eigenvectors of L should be? Check your answer by writing the matrix representation of L with respect to the standard basis $\{1, i\}$, then finding the eigenvalues and eigenspaces of the matrix representation.

$L(1) = i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad$ $L(i) = i^2 = -1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

so $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\quad$ $p(\lambda) = \lambda^2 - 0\lambda + 1 = \lambda^2 + 1$

$\lambda = \pm i$

$\lambda = -i$: $\left(\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right)$ $\quad$ $x_1 = \frac{1}{i}t = -it$, $x_2 = t$ $\quad$ $\bar{x} = \begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$\lambda = i$ .. $\quad$ $\bar{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

6. Solve the initial value problem

$$\begin{cases} y_1'' &= -2y_2 + y_1' + 2y_2' \\ y_2'' &= 2y_1 + 2y_1' - y_2' \\ & y_1(0) = 0, y_2(0) = 0, \\ & y_1'(0) = -3, y_2'(0) = 2. \end{cases}$$

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ y_1' \\ y_2' \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 1 & 2 \\ 2 & 0 & 2 & -1 \end{pmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ 0 & -2 & 1-\lambda & 2 \\ 2 & 0 & 2 & -1-\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 0 & 1 \\ -2 & 1-\lambda & 2 \\ 0 & 2 & -1-\lambda \end{vmatrix} + 1 \begin{vmatrix} 0 & -\lambda & 1 \\ 0 & -2 & 2 \\ 2 & 0 & -1-\lambda \end{vmatrix}$$

$$= -\lambda \left[-\lambda \begin{vmatrix} 1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} + 1 \begin{vmatrix} -2 & 1-\lambda \\ 0 & 2 \end{vmatrix} \right] + 1 \cdot 2 \begin{vmatrix} -\lambda & 1 \\ -2 & 2 \end{vmatrix}$$

$$= -\lambda \left(-\lambda \left((1-\lambda)(-1-\lambda) - 4 \right) + 1(-4) \right) + 2 \left(-2\lambda + 2 \right)$$

$$= -\lambda \left(-\lambda (\lambda^2 - 1) + 4\lambda - 4 \right) - 4\lambda + 4$$

$$= \lambda^2 (\lambda^2 - 1) - 4\lambda^2 \cancel{+4\lambda} \cancel{-4\lambda} + 4$$

$$= \lambda^4 - \lambda^2 - 4\lambda^2 + 4 = (\lambda^2)^2 - 5\lambda^2 + 4 = (\lambda^2 - 4)(\lambda^2 - 1)$$

$$\lambda = \pm 2, \pm 1$$

$$\lambda_1 = 1 \quad \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & -2 & 0 & 2 \\ 2 & 0 & 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \bar{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}$$

$$\lambda_2 = -1: \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -2 & 2 & 2 \\ 2 & 0 & 2 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \bar{x}_2 = \begin{pmatrix} 2 \\ -1 \\ -2 \\ 1 \end{pmatrix}$$

$$\lambda_3 = 2: \begin{pmatrix} -2 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ 0 & -2 & -1 & 2 \\ 2 & 0 & 2 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1/2 & 0 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \bar{x}_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}$$

$$\lambda_4 = -2: \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & -2 & 3 & 2 \\ 2 & 0 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \bar{x}_4 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ -2 \end{pmatrix}$$

$$X = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 1 \\ 1 & -2 & 2 & 2 \\ 2 & 1 & 2 & -2 \end{pmatrix} \qquad X^{-1} = \frac{1}{9} \begin{pmatrix} 0 & 6 & -3 & 0 \\ 3 & -1 & -1 & 0 \\ 0 & -5 & 4 & 3 \\ 3 & 1 & 1 & -3 \end{pmatrix} \qquad \bar{c} = X^{-1} Y_0 = \frac{1}{9} \begin{pmatrix} 0 & 6 & -3 & 0 \\ 3 & -1 & -1 & 0 \\ 0 & -5 & 4 & 3 \\ 3 & 1 & 1 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -3 \\ 2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 3 \\ 1 \\ -2 \\ -3 \end{pmatrix}$$

So,

$$Y = X e^{tD} \bar{c} = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 1 \\ 1 & -2 & 2 & 2 \\ 2 & 1 & 2 & -2 \end{pmatrix} \begin{pmatrix} e^{t} & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & 0 \\ 0 & 0 & e^{2t} & 0 \\ 0 & 0 & 0 & e^{-2t} \end{pmatrix} \begin{pmatrix} 1 \\ 1/3 \\ -2/3 \\ -1 \end{pmatrix} \Rightarrow \begin{cases} y_1 = e^{t} + \frac{2}{3} e^{-t} - \frac{2}{3} e^{2t} - e^{-2t} \\ y_2 = 2e^{t} - \frac{1}{3} e^{-t} - \frac{2}{3} e^{2t} - e^{-2t} \end{cases}$$