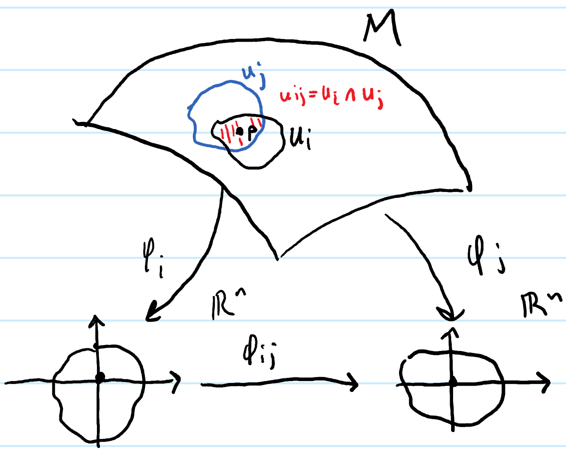
Hausdorff space – topological space in which distinct points have disjoint neighborhoods.

<u>Manifolds</u> (P. Parker)



(u_i, φ_i) chart at p

$\varphi_{ij} = \varphi_j \circ \varphi_i^{-1} : \varphi_i(u_{ij}) \to \varphi_j(u_{ij})$

homeomorphism

<u>Assumptions</u>

manifold : ① T_2 (Hausdorff)

② Locally Euclidean

($\Rightarrow$ around each pt there is a set homeomorphic to $\mathbb{R}^n$)

③ Paracompact. (look up)

("Need" this for partitions of unity.)

A smooth manifold requires that all transition functions are $C^\infty(\mathbb{R}^n, \mathbb{R}^n)$

A covering of charts $\mathcal{U} = \{(u_i, \varphi_i)\}_i$ is an Atlas.

A maximal atlas is called a <u>differential structure</u> on M.

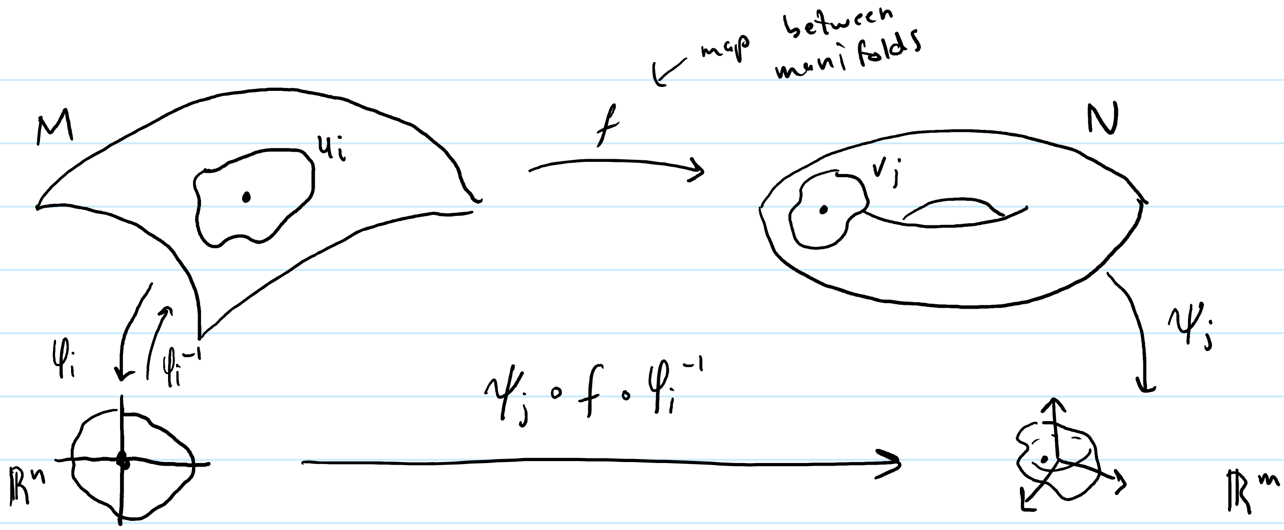
<u>Ex</u>. $\mathbb{R}^n, n \neq 4$, has a unique differential structure.

$\mathbb{R}^4$ has infinitely many distinct differential structures.

<u>Ex</u>. S^7 has 28 distinct diff. structures.

<u>Ex</u>. $(\mathbb{R}^n, Id_n)$ is a manifold. (atlas w/ one chart)

$\mathbb{R}$ is a manifold.



$\psi_j \circ f \circ \varphi_i^{-1}$ is a local representation of f.

$f_{ij} : \mathbb{R}^n \to \mathbb{R}^m$

<u>Defn</u>: $f : M \to N$ is smooth iff each f_{ij} is smooth.

<u>2.4 The function algebra</u>

Let $C(M) = C(M, \mathbb{R})$ be the space of cont. functions on M, and $C^\infty(M)$ the smooth functions.

$$C^\infty(M) \subseteq C(M)$$

we write $\mathcal{F}(M) = \mathcal{F} := C^\infty(M)$

Giving (or choosing) $\mathcal{F}(M) \subseteq C(M)$ is equivalent to specifying the differential structure.

$\mathcal{F}$ is a subalgebra of $C(M)$. The algebra operations are all pointwise.

Thm. $C^\infty = \mathcal{F}$ is dense in $C(M)$

$\Rightarrow$ we can approx. any cont. function w/ smooth ones.

Thm. $C^\infty(M,N)$ is uniformly dense in $C(M,N)$

Thm. M is diffeomorphic to $\tilde{M}$ iff $\mathcal{F}$ is isomorphic to $\mathcal{F}(\tilde{M})$.

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diffeomorphism: $f : M \to \tilde{M}$ w/ f^{-1} and f^{-1} is smooth.

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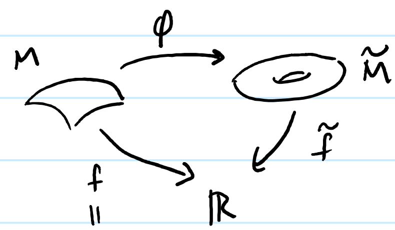
from the proof. $\varphi : M \to \tilde{M}$ smooth then

$\varphi^* : \mathcal{F}_{\tilde{M}} \to \mathcal{F}_M : \tilde{f} \mapsto \tilde{f} \circ \varphi$



$f = \tilde{f} \circ \varphi = \varphi^* \tilde{f}$

So $\mathcal{F}$ is a contravariant functor (cofunctor) from M_f to $\mathbb{R}$-Alg.

(changes order)

2.5 Derivations

Def'n. A Lie algebra is K-module (K is commutative ring w/ iden.) L w/ a product $[\,,\,] : L \times L \to L$ satisfying

(module is a vector space on a Ring.)

① bilinearity: $[\alpha x + \beta y, z] = \alpha[x,z] + \beta[y,z]$ (and in slot z)

② alternating: $[x,x] = 0$ for $x \in L$

③ Jacobi: $[[x,y],z] + [[z,x],y] + [[y,z],x] = 0$

$x, y, z \in \mathbb{R}$

We can let $\mathbb{K}$-module be a real vector space. Then we have

2^*. <u>Skew-symm</u>. $[x,y] = -[y,x]$ (alternating $\Rightarrow$ skew)

<u>Ex</u>. $\mathbb{R}^3$ w/ the cross product

<u>Ex</u>. $\mathbb{R}^n$ w/ $[,] = 0$.

<u>Ex</u>. Heisenberg Algebra ($\mathbb{R}^3$)

$\{x, y, z\}$ w/ $[x,y] = z$ and $[y,x] = -z$

<u>Ex</u>. $\mathfrak{gl}_n = \mathbb{R}^{n \times n}$ ($n \times n$ matrices) w/ $[A,B] = AB - BA$

<u>Def'n</u>. Let $\mathfrak{g}$ be a lie algebra. A derivation of $\mathfrak{g}$ is a map $D: \mathfrak{g} \to \mathfrak{g}$ satisfying

$$D[x,y] = [Dx, y] + [x, Dy]$$

In general, let A be an $\mathbb{R}$-algebra. A derivation of A is a map $D: A \to A$ obeying $D(fg) = D(f)g + f D(g)$ (Leibniz Rule).

Let M be a smooth manifold w/ function algebra $\mathcal{F}$. Let $p \in M$ and let (U, x) is a chart at p ($\Rightarrow x(p) = (0, \dots, 0)$ and $x = (x^1, x^2, \dots, x^n)$)
(x is the chart map. gives us coordinates.)

Let $\mathrm{Der}(\mathcal{F})$ = derivations of $\mathcal{F}$. It turns out that the partial derivatives $\{\partial/\partial x^1, \partial/\partial x^2, \partial/\partial x^3, \dots, \partial/\partial x^n\} = \{\partial_1, \partial_2, \dots, \partial_n\}$ form a basis for $\mathrm{Der}(\mathcal{F})$ at the point p.

$$D = a^1 \partial_1 + a^2 \partial_2 + \dots + a^n \partial_n$$

$\mathrm{Der}(\mathcal{F})_p = T_p M$

Derivations "eat" functions. In a coordinate chart, we write

$$Df(p) = a^1(p)\,\partial_1 f(p) + a^2(p)\,\partial_2 f(p) + \dots + a^n(p)\,\partial_n f(p)$$

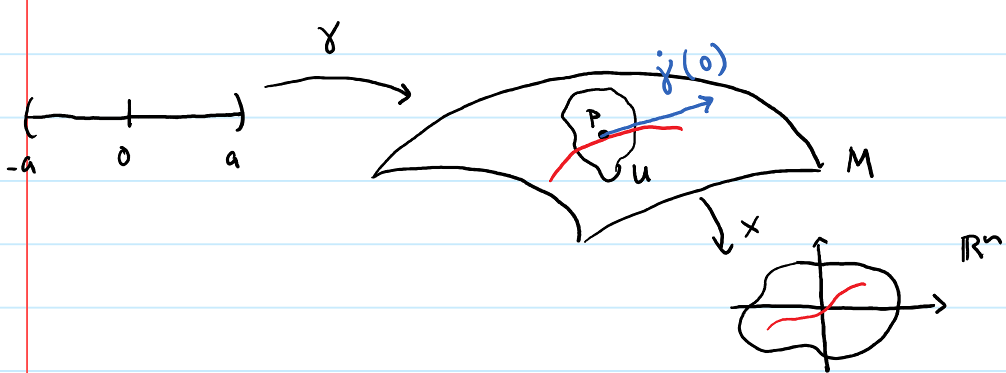
$D_p = a^1(p)\partial_1 + \dots + a^n(p)\partial_n$ (member of the vector space $\mathrm{Der}(\mathcal{F})_p$)

$D_p \in \mathrm{Der}(\mathcal{F})_p = \begin{pmatrix} a^1 \\ a^2 \\ \vdots \\ a^n \end{pmatrix}(p)$

The <u>tangent space</u> to M at p is $T_pM := \mathrm{Der}(\mathcal{F})_p \cong \mathbb{R}^n$

Tangent vectors "act like" derivatives

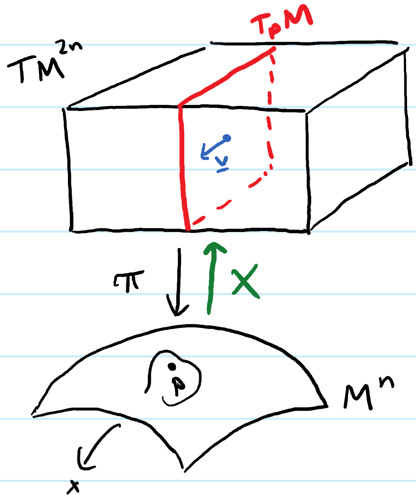
Another point of view:



$(x \circ \gamma')(0)$ is a vector in the Calc. 3 sense that is tangent to the curve at the origin.

collecting all the tangent vectors, you get the tangent space T_pM.
Think of the tangent plane (copy of $\mathbb{R}^n$).

Tangent bundle : $TM = \bigcup_{p \in M} T_pM$ (disjoint union by tangent planes do not interact w/ eachother.)



$\pi : TM \longrightarrow M$

$: \underline{v} \longmapsto p$

A (smooth) section of TM is a smooth map $X : M \longrightarrow TM$ such that $\pi \circ X = Id$. ("right inverse")

X assigns a vector to every point in M. The vectors change continuously. As it picks different points you get a curve in TM.

A smooth section of TM is a <u>vector field</u> on M.

The space of all vector fields is $\mathfrak{X}(M)$.

$\mathfrak{X}(M)$ is a Lie algebra w/ commutator bracket

$[X, Y] = XY - YX$ is a v.f. !

In local coordinates, the vector field X can be written

$$X = a^1 \partial_1 + a^2 \partial_2 + \dots + a^n \partial_n$$

So a v.f. is a derivation (i.e. "eats" functions)

Thursday, January 26, 2017 2:58 PM

Thus Xf makes sense. " $X : \mathcal{F} \to \mathcal{F}$ "

H.W. Prove $[X,Y] = XY - YX$ is a vector field.

Let $X = \partial_i$, $Y = \partial_j$

$$[\partial_i, \partial_j] f = \partial_i\big(\partial_j(f)\big) - \partial_j\big(\partial_i(f)\big)$$