Fiber bundles

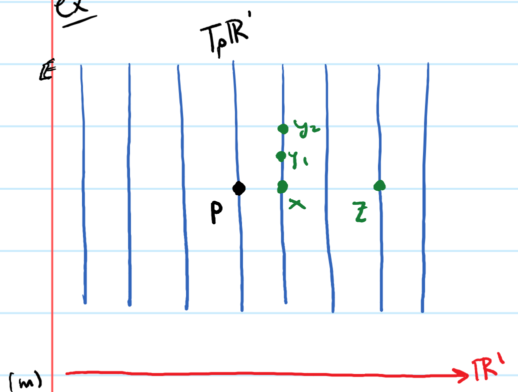
Let E, F and M be smooth manifolds and let $\pi : E \twoheadrightarrow M$ be a smooth submersion (π_* is a surjection (π_* = tangent map)), such that the $\pi^{-1}(p) \cong F$ for each $p \in M$.

(submersion – it is a surjection and the tangent map is a surjection.)

$$\begin{array}{ccc} F & \hookrightarrow & E \\ & & \downarrow \pi \\ & & M \end{array}$$

E = total space (bundle)
F = model fiber
M = base manifold
$\pi^{-1}(p)$ = fiber over p.

ex

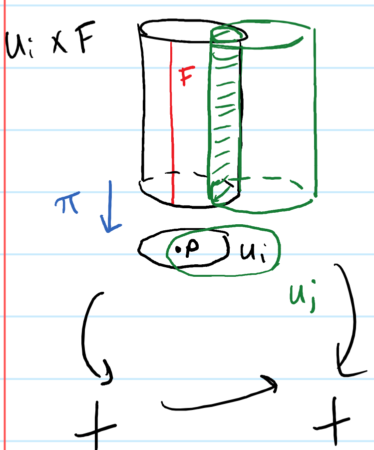


$T\mathbb{R}^1 = \mathbb{R}^1 \times \mathbb{R}^1 = \mathbb{R}^2$

allowed $(x, y_1) + (x, y_2) = (x, y_1 + y_2)$

cannot $(x, y_1) + (z, y_1)$? ✗

$\forall p$, $\exists$ nbhd U_i w/ $p \in U_i$ such that we can let U_i be a chart in M. Since M is paracompact, we get <u>local</u> <u>triviality</u> : $\pi^{-1}(U_i) \cong U_i \times F$



can call $\varphi_i : \pi^{-1}(U_i) \cong U_i \times F$

Set $U_{ij} = U_i \cap U_j$, and define
$\varphi_{ij} : \varphi_i \circ \varphi_j^{-1} : U_{ij} \times F \to U_{ij} \times F$

WLOG $\varphi_{ij}|_{U_{ij}} = Id|_{U_{ij}}$ for fixed
so $\varphi_{ij}(p, v) = \varphi_{ij}(p, \Phi(v))$. Then if we fix $p \in U_{ij}$, then $\varphi_{ij} = \varphi_{ij}(p, \ ) : F \to F$. $\varphi_{ij} \in Aut(F)$.

Aut (F) is a Lie group: group <u>and</u> manifold, Group operations are smooth.

The collection of all $\{U_{ij}\}$ are the <u>bundle</u> <u>cocycle</u>.

The cocycle condition is: ii) $\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik}$

this implies i) $\varphi_{ij}^{-1} = \varphi_{ji}$

cocycles determine the structure of the bundle, and vice-versa.

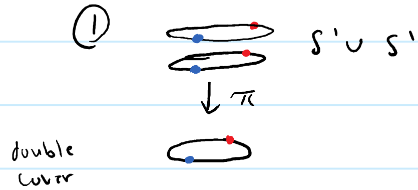
$\{\psi_{ij}\} \sim \{\varphi_{ij}\}$ if $\exists$ family $\{h_i \in Aut(F)\}$ s.t. $\psi_{ij} = h_i^{-1} \circ \varphi_{ij} \circ h_j$

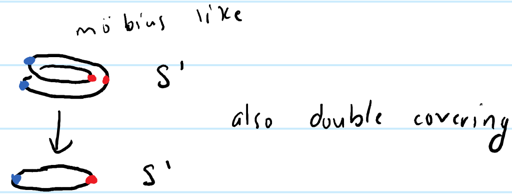
<u>Def'n</u>. A bundle <u>morphism</u> is a pair (u, f) of smooth maps $u: E \to E'$ and $f: M \to M'$ s.t.

commutes
$$\begin{tikzcd} E \arrow[r, "u"] \arrow[d, "\pi"'] & E \arrow[d, "\pi'"] \\ M \arrow[r, "f"] & M' \end{tikzcd}$$
u preserves fibers and completely determines f.

<u>Ex</u>. A <u>covering space</u> is a fiber bundle w/ discrete fibers.

Let $M = S^1$ ①





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<u>If</u> $F = V$ is a vector space, then $\pi : E \xrightarrow{F} M$ is a vector bundle.

WLOG, $F = \mathbb{R}^k$.

The tangent bundle $TM = \bigcup_p T_pM$; $T_pM \cong F \cong \mathbb{R}^n$, so TM is a vector bundle.

(fibers; dim. of manifold)

<u>Ex.</u> Let $f : M \to N$ be a smooth map of manifolds, and $\pi : E \to N$ a fiber bundle.

$$\begin{tikzcd}
f^*E \arrow[r, "f_\natural"] \arrow[d, "f^*E"'] & E \arrow[d, "\pi"] & F \arrow[l] \\
M \arrow[r, "f"'] & N
\end{tikzcd}$$

$\pi^{-1}(f(p)) \cong F$ (model)

Pull fiber back through f.

$M \ni p$, $N \ni f(p)$

f^* is a functor

$\natural$ - \natural in Latex

The fibers of f^*E are defined to be $(f^*E)_p := E_{f(p)}$

The Vertical Bundle

Let $\pi : E \to M$ be a fiber bundle, w/ M^n (means it locally look like $\mathbb{R}^n$) and F^k (same idea) so $\dim(E) = n + k$

$$\begin{tikzcd}
TE \arrow[r, "\pi_* \text{ (Jacobian on each fiber)}"] \arrow[d, "\pi_T"'] & TM \arrow[d, "\pi_M"] \\
E \arrow[r, "\pi"'] & M
\end{tikzcd}$$

$E \ni v$, $M \ni p$

$\pi_* = d\pi = T\pi$

Since π is a submersion, π_* is surjective (onto). If $\pi(v) = p$, then

$$\begin{array}{ccc} \pi_* : T_vE & \to & T_pM \\ \cong & & \cong \\ \mathbb{R}^{n+k} & \to & \mathbb{R}^n \end{array}$$

The kernel of π_* is the <u>vertical</u> <u>bundle</u> of dim. k. : $\mathcal{V} := \ker(\pi_*)$

The $\mathcal{V}_v$ (single fiber in TE above v.) consist of the vectors in T_vE that are tangent to the fiber E_p.