homeomorphism : cont. funct. w/ cont. inverse
discrete top. : each pt is open + closed.

Def'n. A <u>sheaf</u> on a topological space X is a topological space $\mathcal{S}$ called the *étalé* space or sheaf space, together with a continuous, surjective, local homeomorphism

$\pi : \mathcal{S} \to X$ such that

① each stalk $\mathcal{S}_x := \pi^{-1}(x)$ is an algebraic subject object (group, ring, K-module) w/ the discrete topology.

② All operations are continuous.

$$\times : \mathcal{S}_x \times_\pi \mathcal{S}_x \longrightarrow \mathcal{S}_x \quad \text{w/} \quad \mathcal{S}_x \times_\pi \mathcal{S}_x := \{ (s_1, s_2) \in \mathcal{S} \times \mathcal{S} \mid \pi(s_1) = \pi(s_2) \}$$

Idea is that we attach an algebraic structure to each point in a topological space (like a manifold).

Def'n.

$$\begin{tikzcd} \mathcal{S} \arrow[r, "u"] \arrow[d, "\pi"'] & \mathcal{F} \arrow[d, "\pi'"] \\ X \arrow[r, "f"] & Y \\ x \arrow[u, phantom, "\in" sloped] & f(x) \arrow[u, phantom, "\in" sloped] \end{tikzcd}$$

u preserves stalks $u : \mathcal{S}_x \to \mathcal{F}_{f(x)}$ (in the sense that it maps stalk to stalk.)
and u is a morphism of the algebraic structure.

u completely determines f.

<u>Ex</u> Any continuous right inverse to π, $\pi \circ \sigma = id_X$, is a <u>section</u> of $\pi : \mathcal{S} \longrightarrow X$.

The space of all sections of $\mathcal{S}$ is denoted by $\Gamma(\mathcal{S})$ or $\Gamma(X, \mathcal{S})$

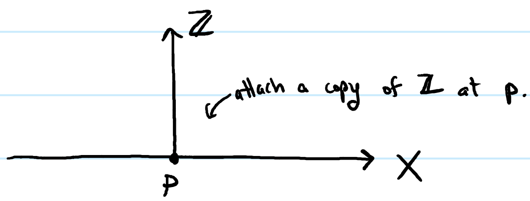
$(X, \mathcal{S})$ is a map from $\underline{\mathcal{S} \to X}$.

Is the requirement of continuity in pt ② of the def. of a sheaf necessary if we always use the discrete topology?

The section space is the same type of algebraic object as the stalks (under pt-wise ops).

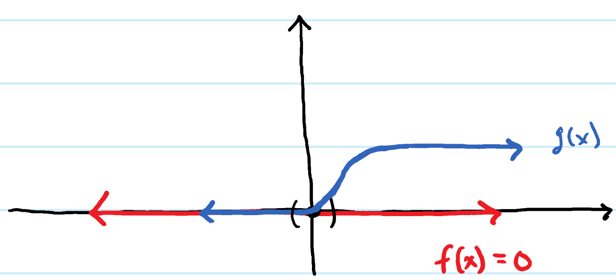
Ex. Skyscraper Sheaf

X, $p \in X$. Define $\mathcal{S}_p = \mathbb{Z}$ and $\mathcal{S}_q = \{0\}$ at each point $q \neq p$.



Def'n. Let X be a topological space. $C(X) = \{$ cont. fncts $f: X \to \mathbb{R} \}$.

Two functions f and g are germ-equivalent at $x \in X$ iff $\exists$ an open nbhd U, w/ $x \in U$ such that $f|_U = g|_U$.



$f = g$ for $x < 0$

$f(0) = g(0)$

$f(x) \neq g(x)$ for $x > 0$.

These are not germ-equivalent at $x = 0$ b/c they are not equal on any nbhd where $x > 0$.

An equivalence class $[f]_x$ of this relation is called a germ at x.

Choose a function f, then $[f]$ is its germ as a function on X.

sheaf of cont. functions on X.

equiv. classes

Ex let $\mathcal{C}_x$ is the union of all germs at $x \in X$ and consider the set

$$\mathcal{C}(X) = \bigcup_{x \in X} \mathcal{C}_x \qquad \text{define } \pi : \mathcal{C}(X) \to X \text{ by } [f]_x \mapsto x$$

Let $\mathcal{U} \subset X$ be open, and let $f \in C(\mathcal{U})$. From sets $\bigcup_{x \in \mathcal{U}} [f]_x \subseteq \mathcal{C}(x)$

These sets form a base for the topology on $\mathcal{C}(x)$.

Thus, $\mathcal{C}(x)$ is a sheaf over X called the <u>sheaf of germs of cont. functions on X</u>.

$\mathcal{C}^\infty$ = "smooth" when X is a manifold and $f \in C^\infty$

$\xi : E \underset{\sigma}{\rightleftarrows} M$ a fiber bundle, $\Gamma(E) = \{ \sigma : M \to E \text{ smooth} \mid \xi \circ \sigma = id_M \}$

Then define $\mathcal{E} \to M$ to be the sheaf of germs of smooth sections of E.

In particular $\mathcal{C}$ is not Hausdorff! (i.e. e.g. $\mathcal{E}$ cannot be a manifold)

Taking limits means we lose Hausdorff in these types of spaces.

$$\begin{array}{c} \mathcal{C}(x) \\ \pi \big\downarrow \big\uparrow \sigma \\ X \end{array}$$

$\sigma : X \to \mathcal{C}(x)$ is a cont. section.

$\Gamma(\mathcal{C}(x)) = \{ \text{section space} \}$

<u>Theorem</u>. The $\mathbb{R}$-algebras $C(X)$ and $\Gamma(\mathcal{C}(x))$ are isomorphic.

$f_x \mapsto [f]_x$ pt. wise.

Def'n. Let X be a top. space and let $\underline{C}$ be a category. (usually $\underline{C}$ = Set, Ring, Grp, ...). A presheaf $\mathcal{F}$ on X is a functor w/ values on $\underline{C}$ given to the following data:

① for each $U \subseteq X$ open, there corresponds on $\mathcal{F}(U) \in \underline{C}$.

② For each inclusion $V \subseteq U$, there is a morphism $\rho_{VU} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ in $\underline{C}$.

(ex. if category is groups, then ρ is a morphism).

$\mathcal{F} : \mathcal{O}(X) \longrightarrow \underline{C}$, $\mathcal{O}(X)$ is the category of open sets of X, w/ inclusion maps as arrows.

$\mathcal{F}$ attaches an object (categorical object) to each open set.

For each $U \in \mathcal{O}(X)$, the elements of $\mathcal{F}(U)$ are the sections of $\mathcal{F}$ over U.

Def'n. A presheaf $\mathcal{F}$ on X is a sheaf if it satisfies the locality and gluing properties.

locality: If $\{U_i\}$ is on open cover of V and if $s, t \in \mathcal{F}(V)$ such that $s|_{U_i} = t|_{U_i}$ for all i, then $s = t$.

gluing: If $\{U_i\}$ is an open cover of V; and if for each i, $s_i \in \mathcal{F}(U_i)$ s.t. for each pair (U_i, U_j) $s_i = s_j$ on $U_{ij} = U_i \cap U_j$, then there is an $s \in \mathcal{F}(V)$ s.t. $s|_{U_i} = s_i$ and $s|_{U_j} = s_j$.

Sheaves on Manifold

$\mathcal{C}^j \to M$ (germs of) j-times diff'able functions on M

$\Omega^p \to M$ sections are differential p-forms on open U.

"cotangent sheaf" stalks are groups under pt. wise mult.

$\mathcal{D} \to M$ sections are finite-order diff. ops. on open U.

stalks are

A pair $(X, \mathcal{O}_X)$ where $\mathcal{O}_X$ is a sheaf of rings on X, is called a ringed space.

A ringed space - sheaf where all stalks are rings.

An important case: each stalk is a local ring: a ring w/ unique maximal ideal $\mathfrak{m}$. This is called a locally ringed space.

Ex An n-dim C^∞-manifold M is a locally ringed space whose sheaf $\mathcal{O}_M$ is isomorphic to the sheaf of smooth functions on $\mathbb{R}^n$. Stalks are $\mathbb{R}$-algebras.