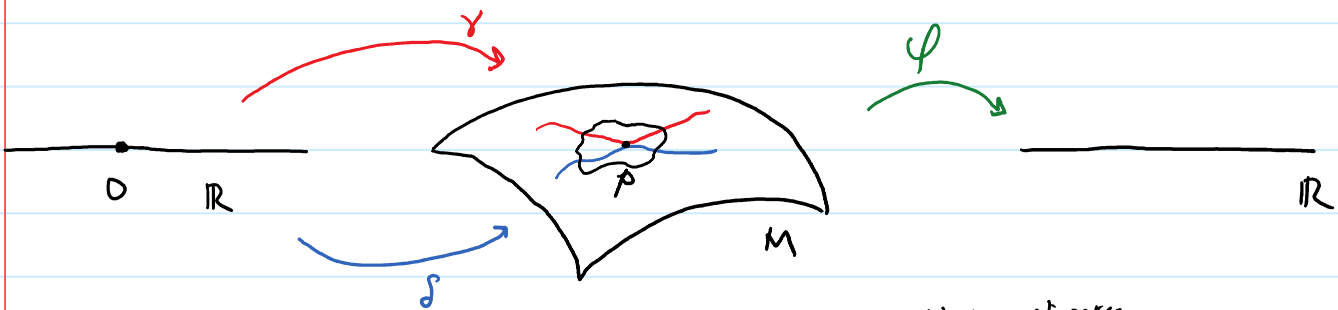
Jets following KMS Ch. 4.

Defn'. Let $\gamma, \delta : \mathbb{R} \to M$ smooth curves. γ and δ have ***r^th^ order contact*** at $0 \in \mathbb{R}$ iff for every smooth function $\varphi \in \mathcal{S}M$, then $\varphi \circ \gamma - \varphi \circ \delta$ vanishes to r^{th} order at $0 \in \mathbb{R}$



$\varphi \circ \gamma : \mathbb{R} \to \mathbb{R}$ where $(\varphi \circ \gamma)^{(k)}(0) = (\varphi \circ \delta)^{(k)}(0)$ (all derivatives must agree.)

$\varphi \circ \delta : \mathbb{R} \to \mathbb{R}$ for all $0 \leq k \leq r$

Prop. $\gamma \sim_r \delta$ is an equiv. relation. $\square$ do this.

- 0-order contact $\Rightarrow$ the two curves intersect at a point.
- contact only depends on the germs of γ, δ.

Lemma. $f : M \to N$ smooth map between manifolds. If $\gamma \sim_r \delta$, then
$$(f \circ \gamma) \sim_r (f \circ \delta)$$
i.e. composition w/ a smooth map perserves contact.

Defn'. Two smooth maps $f, g : M \to N$ are *r-jet equivalent* at a point $p \in M$, iff for every smooth curve $\gamma : \mathbb{R} \to M$ s.t. $\gamma(0) = p$, then the curves $(f \circ \gamma)$ and $(g \circ \gamma)$ in N have r^{th}-order contact at $0 \in \mathbb{R}$.

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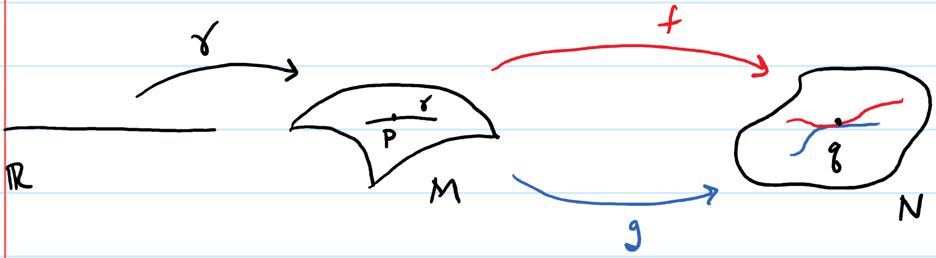

stuff agrees in some neighborhood.

we write $j^r_p f = j^r_p g$ or $j^r f(p) = j^r g(p)$. p is in domain

an r-jet is an equivalence class under this relation.

- jets defined at a point.
- r-jets depend only on the <u>germ</u>.

- The set of all r-jets denoted by $J^r(M,N)$ (jets can be local maps)

If $X \in J^r(M,N)$, then we can find an f (pointwise) s.t.
$$X = j^r_p f.$$

The <u>source map</u> is $\sigma(X) := p \in M$
The <u>target map</u> is $\tau(X) := f(p) = q \in N$

source and target are unique.

$J^r_p(M,N)$ = set of all r-jets with source p.
$J^r(M,N)_q$ = set of all r-jets with target q.

$J^r_p(M,N)_q := J^r_p(M,N) \cap J^r(M,N)_q$ - set of all r-jets w/ source p and target q.

Denote by π^r_s, $0 \le s \le r$, the projection $\pi^r_s : j^r_p f \longmapsto j^s_p f$
projection to a lower-order jet space.

All r-jets form a category, the units of which are identity maps of manifolds.

(unit means all arrows are invertable.)

An $X \in J_p^r(M,N)_q$ is <u>invertible</u> iff $\exists$ $X^{-1} \in J_q^r(N,M)_p$ such that

$$X^{-1} \circ X = j_p^r(id_M)$$

$$X \circ X^{-1} = j_q^r(id_N)$$

<u>Im FT</u>: $X \in J^r(M,N)$ is invertible iff $\pi_1^r(X) \in J^1(M,N)$ is invertible.

$\text{inv}\, J^r(M,N)$

<u>Phil Parker - local coord. POV</u> -

<u>Def'n</u>. A <u>multi-index</u> of <u>range</u> n is an n-tuple of non-negative integers $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$.

$$|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$$
$$\alpha! = \alpha_1!\,\alpha_2! \ldots \alpha_n! \qquad (0! = 1)$$

let $x = (x^1, x^2, \ldots, x^n)$ ← component fnct. $\in \mathbb{R}^n$

then $x^\alpha = (x^1)^{\alpha_1} \cdot (x^2)^{\alpha_2} \cdot \ldots \cdot (x^n)^{\alpha_n}$

$$D_\alpha f = \frac{\partial^{|\alpha|} f}{(\partial x^1)^{\alpha_1} (\partial x^2)^{\alpha_2} \cdots (\partial x^n)^{\alpha_n}}$$

the partial derivative of $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$.

If $|\alpha| = k$, the multinomial coefficients are

$$\binom{k}{\alpha} := \frac{k!}{\alpha!} = \frac{k!}{\alpha_1! \alpha_2! \cdots \alpha_n!}$$

The multinomial expansion is

$$(x^1 + x^2 + x^3 + \dots + x^n)^k = \sum_{|\alpha| = k} \binom{k}{\alpha} x^\alpha$$

Defn. The MacLaurin-Taylor polynomial of order r of $f \in \mathcal{F}(U)$ at 0 is

$$p^r f = \sum_{k=0}^{r} \sum_{|\alpha|=k} \binom{k}{\alpha} D_\alpha f(0)\, x^\alpha$$

the r^{th}-order polynomial approximation of f at p.

Defn! Two functions $f, g \in \mathcal{F}(U)$ are r-jet equivalent at p iff $p^r f = p^r g$ at p.

RE. This is independent of choice of local coordinates. (Prove This.)

Back to Bundles

Let $\pi : E \to M$ be a smooth fiber bundle w/ $\dim M = n$ and $\dim F = k$. ($F = \pi^{-1}(p)$ to make model fiber).

$$J^r E = J^r(\pi) = J^r(E \xrightarrow{\pi} M)$$

= set of r-jets of local sections of π.

$\{ \sigma : U \subset M \longrightarrow E \mid \pi \circ \sigma = id_M \}$

$J^r E$ is the r^{th}-jet prolongation.

$J^r E \subset J^r(M, E)$ is a closed submanifold.

Two bundle structures:

① $\sigma : J^r E \to M$ (bundle over M where all the fibers are the jets whose source is E. ($J^r_p E$)

② $\tau : J^r E \to M$ (bundle over fibers are along fibers of E. $(J^r E)_q$

$J^0 E = E$

we still have $\pi^r_s : J^r E \to J^s E$

- $\tau : J^r E \to M$ is always an affine bundle for every E.

(affine - vector space w/o origin)

- When E is a vector bundle, then $J^r E$ is a vector bundle.

We obtain a <u>tower</u> of jet prolongations.

$$\begin{tikzcd}
\vdots \arrow[d] \\
J^3E \arrow[d, "\pi_2^3"] \\
J^2E \arrow[d, "\pi_1^2"] \\
J^1E \arrow[d, "\pi_0^1 = \tau"] \\
E \arrow[d, "\pi"] \\
M
\end{tikzcd}$$

The <u>infinite jets</u> are elements of
$$J^\infty E = \varprojlim J^r E$$

$J^\infty E$ is <u>not</u> a manifold, hence not a bundle. But it is a <u>ringed space</u>! (sheafs!)